Paper II : Differential Equations

Unit - II

Linear Differential Equation of second order & their solutions.

Diff. Eq.^{n} of the form

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R \quad \text{——— (i)}$$

$\underbrace{P,\ Q,\ R}$

funⁿ of x (or constant in special case)

is known as linear differential equation of second order.

There is no generalised method.

E.

(*) The method of finding an integral of the C.F. by inspection :-

C.F. → पूरक – फलन

Table – I

	condition	An integral of CF (solution)
1.	$1 + P + Q = 0$	$y = e^{x}$
2.	$1 - P + Q = 0$	$y = e^{-x}$
3	$1 + \frac{P}{m} + \frac{Q}{m^2} = 0$	$y = e^{mx}$
4	$P + Qx = 0$	$y = x$
5	$2 + 2Px + Qx^2 = 0$	$y = x^2$
6	$m(m-1) + Pmx + Qx^2 = 0$	$y = x^m$

Working procedure :-

Step 1 : put the given equation in the standard form $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$ — (1)

~~ke~~ keeping the coeffecient of $\frac{d^2y}{dx^2}$ is unity.

Step 2 : Find an integral of CF with the help of Table -I.

If a solution $\underline{u}$ is given, then skip this step.

Step 3 : Put $\boxed{y = uv}$ in given DE (1) and simplify it to

[let this ue CS of (1)]

memorize.

$$\boxed{\frac{d^2v}{dx^2} + \left(P + \frac{2}{u}\frac{du}{dx}\right)\frac{dv}{dx} = \frac{R}{u}} \quad \text{— (2)}$$

Step 4 : Take $\frac{dv}{dx} = p$ in eq.^{n} (2) and reduce it to linear DE in p.

$\frac{dy}{dx} + Py = Q \longrightarrow$ I.F. $= e^{\int P dx} \longrightarrow$ sol^n is $y(I.F.) = \int Q(I.F.)dx + c$

Linear DE

by using above method, determine value of $p \left(= \frac{dv}{dx}\right)$

Step 5 : Determine the value of v by integration.

Step 6 : Hence $y = uv$ is CS of LDE (1)

Ex:- 1 $\quad \dfrac{d^2y}{dx^2} - x^2\dfrac{dy}{dx} + xy = x.$

Sol^n:→ I) $\dfrac{d^2y}{dx^2} + (-x^2)\dfrac{dy}{dx} + x\, y = x$ —(i)

$\uparrow P \qquad \uparrow Q \qquad \uparrow R$

(II) Since $P + Qx = 0 \Rightarrow y = \underbrace{x}_{u}$ is a part or integral of CF.

(III) put $\boxed{y = uv = xv}$ in DE (i), we have
↳CS

$$\frac{d^2v}{dx^2} + \left[(-x^2) + \frac{2}{x}\cdot 1\right]\frac{dv}{dx} = \frac{x}{x} \quad \text{(Eq.}^n \text{ (2) of working procedure)}$$

$$\frac{d^2v}{dx^2} + \left[-x^2 + \frac{2}{x}\right]\frac{dv}{dx} = 1 \quad \text{—(ii)}$$

(IV) put $\dfrac{dv}{dx} = p$ in (ii)

$$\frac{dp}{dx} + \left[-x^2 + \frac{2}{x}\right]p = 1 \text{ —(iii)} \quad \text{((LDE) of first order)}$$

$$\text{I.F.} = e^{\int\left(-x^2 + \frac{2}{x}\right)dx} = e^{-\frac{x^3}{3} + 2\log x} = e^{-\frac{x^3}{3}} e^{\log x^2}$$
$$= x^2 e^{-\frac{x^3}{3}}$$

Hence solution of Eq.^{n} (iii) is

$$p.(\text{I.f.}) = \int 1.(\text{I.f.})\,dx + c$$

$$p.\left(x^2 e^{-x^3/3}\right) = \int x^2 e^{-x^3/3}\,dx + c$$

$$= -e^{-x^3/3} + c$$

let $\dfrac{x^3}{3} = t$

$\dfrac{\cancel{3}x^2}{\cancel{3}}dx = dt$

$$p = \frac{dv}{dx} = -\frac{1}{x^2} + \frac{c}{x^2} e^{x^3/3}$$

On integration

$$v = \frac{1}{x} + c\int \frac{1}{x^2} e^{x^3/3}\, dx + c_1$$

Hence $y = uv = x\left[\frac{1}{x} + c\int \frac{1}{x^2} e^{x^3/3}\, dx + c_1\right]$ is the required c.s.

Ex:- $\frac{d^2y}{dx^2} - \cot x \frac{dy}{dx} - (1-\cot x)\, y = e^x \sin x$

sol^n:→ i) $\frac{d^2y}{dx^2} + [-\cot x)\frac{dy}{dx} + (\cot x - 1)\, y = e^x \sin x$ —(i)

$P = -\cot x$, $Q = \cot x - 1$, $R = e^x \sin x$

(ii) $1 + P + Q = 0 \Rightarrow y = e^x$ is an integral of C.F. → u

(iii) Put $\boxed{y = uv = e^x \cdot v}$ in given DE, we have (→ CS)

$$\frac{d^2v}{dx^2} + \left[(-\cot x) + \frac{2}{\cancel{e^x}}\cdot \cancel{e^x}\right]\frac{dv}{dx} = \frac{\cancel{e^x}\sin x}{\cancel{e^x}}$$

$$\frac{d^2v}{dx^2} + [2 - \cot x]\frac{dv}{dx} = \sin x$$

(iv) put $\frac{dv}{dx} = p$ in above Eq.^{n}

$$\frac{dp}{dx} + (2 - \cot x)\, p = \sin x \quad \text{—(ii)}$$

It is LDE in p (first order)

$$I.f. = e^{\int(2-\cot x)dx}$$

$$= e^{2x} \cdot e^{-\log \sin x} = \frac{e^{2x}}{\sin x}$$

Solⁿ. of eqⁿ (ii) is

$$p.\left(\frac{e^{2x}}{\sin x}\right) = \int \cancel{(\sin x)}\left(\frac{e^{2x}}{\cancel{\sin x}}\right)dx + c,$$ where c is constant of integration.

$$p\left(\frac{e^{2x}}{\sin x}\right) = \frac{e^{2x}}{2} + c$$

$$p = \frac{\sin x}{2} + \frac{c \sin x}{e^{2x}}$$

$$p = \frac{dv}{dx} = \frac{1}{2}\sin x + c\, e^{-2x} \sin x$$

On integration

$$v = -\frac{1}{2}\cos x + c\left[\frac{e^{-2x}}{\{(-2)^2+(1)^2\}}(-2\sin x - \cos x)\right] + c_1$$

Here c_1 is constant of integration.

$$v = -\frac{1}{2}\cos x + c\left[\frac{e^{-2x}}{5}(-2\sin x - \cos x)\right] + c_1$$

So $y = uv = -\frac{1}{2}e^{x}\cos x - \frac{1}{5}c\, e^{-x}(2\sin x + \cos x) + c_1 e^{x}$

is the required cs.