Ex:- Solve : $x^2 \frac{d^2y}{dx^2} - 2x(1+x)\frac{dy}{dx} + 2(1+x)y = x^3$

Sol^n:- (i) divide the given DE by x^2 (making the coefficient of $\frac{d^2y}{dx^2}$ as unity)

$$\frac{d^2y}{dx^2} + \underbrace{\left\{-\frac{2(1+x)}{x}\right\}}_{P}\frac{dy}{dx} + \underbrace{\left\{\frac{2(1+x)}{x^2}\right\}}_{Q} y = \underbrace{x}_{R} \quad \text{—(i)}$$

(ii) Since $P + Qx = 0 \Rightarrow y = x$ is an integral of CF.

u

(iii) put $y = uv = xv$ in given DE (i), we have

$$\frac{d^2v}{dx^2} + \left[\frac{-2(1+x)}{x} + \frac{2}{x}\cdot 1\right]\frac{dv}{dx} = \frac{x}{x}.$$

$$\Rightarrow \frac{d^2v}{dx^2} + (-2)\frac{dv}{dx} = 1.$$

(iv) Let $\frac{dv}{dx} = p$, the above DE becomes

$$\frac{dp}{dx} + (-2)p = 1 \quad \text{—(ii)}$$

It is LDE in p (first order)

$$\text{I.F.} = e^{\int(-2)dx} = e^{-2x}$$

Sol^n of Eq.^{n} (ii) is

$$p\cdot(e^{-2x}) = \int 1\cdot(e^{-2x})\,dx + c$$

where c is constant of integration.

$$p = \frac{dv}{dx} = \frac{1}{e^{-2x}}\left[\frac{e^{-2x}}{(-2)} + c\right]$$

$$\frac{dv}{dx} = -\frac{1}{2} + ce^{2x}$$

On integration

$$v = -\frac{1}{2}x + \frac{ce^{2x}}{2} + c_1 \quad (c_1 \to \text{constant of integration})$$

Hence $y = uv = -\frac{x^2}{2} + \frac{cxe^{2x}}{2} + c_1 x$ is required

cs.

Ex:→ Solve $\sin^2 x \frac{d^2y}{dx^2} = 2y$., given that $y = \cot x$ is a solution.

Solⁿ:→ (i) divide the given DE by $\sin^2 x$.

$$\frac{d^2y}{dx^2} + \underbrace{0}_{P} \cdot \frac{dy}{dx} + \underbrace{\left[\frac{-2}{\sin^2 x}\right]}_{Q} y = \underbrace{0}_{R} \quad \text{——— (i)}$$

[no need of step (ii)]

(iii) put $\boxed{y = uv = (\cot x) v}$ $\to$ cs in given DE, (i),

becomes

$$\frac{d^2v}{dx^2} + \left[0 + \frac{2}{\cot x}(-\operatorname{cosec}^2 x)\right]\frac{dv}{dx} = \frac{0}{\cot x}$$

(iv) put $\frac{dv}{dx} = p$ in above, we have.

$$\frac{dp}{dx} + \left\{-\frac{2\operatorname{cosec}^2 x}{\cot x}\right\} p = 0 \quad \text{(ii)}$$

[It is LDE in p (first order)]

$$I.f. = e^{\int \left\{ -\frac{2\,\mathrm{cosec}^2 x}{\cot x} \right\} dx}$$

$$= \exp\left[\int - \frac{2}{\sin^2 x} \frac{\sin x}{\cos x} dx\right]$$

$$= \exp\left[\int - \frac{4}{\sin 2x} dx\right]$$

$$= e^{-\int 4\,\mathrm{cosec}\,2x\,dx}$$

$$= e^{-2 \log \tan x}$$

$$= \frac{1}{\tan^2 x}$$

Hence solⁿ. of eq.ⁿ (ii) is

$$P\left(\frac{1}{\tan^2 x}\right) = \int 0 \cdot \left(\frac{1}{\tan^2 x}\right) dx + c \rightarrow \text{constant of integration}$$

$$P = \frac{dv}{dx} = c \tan^2 x = c(\sec^2 x - 1)$$

On integration

$$v = c(\tan x - x) + c_1$$

where c_1 is the constant of integration.

Hence $y = uv = c \cot x (\tan x - x) + c_1 \cot x)$

is the required CS

Ex:- Solve $(x+2)\frac{d^2y}{dx^2} - (2x+5)\frac{dy}{dx} + 2y = (x+1)e^x$

Solⁿ:→ (i) divide the given DE by $(x+2)$,

$$\frac{d^2y}{dx^2} + \underbrace{\left[-\frac{(2x+5)}{(x+2)}\right]}_{P} \frac{dy}{dx} + \underbrace{\left[\frac{2}{(x+2)}\right]}_{Q} y = \underbrace{\frac{(x+1)e^x}{(x+2)}}_{R} \quad \text{—(i)}$$

(ii) $m^2 + mP + Q = 0$ (to determine the value of m)

$\Rightarrow m^2 + \left[-\frac{(2x+5)}{(x+2)}\right] m + \frac{2}{x+2} = 0$

$\Rightarrow m^2(x+2) - (2x+5)m + 2 = 0$

$\Rightarrow m^2x - 2xm + 2m^2 - 4m + 2 - m = 0$

$\Rightarrow mx(m-2) + 2m(m-2) - 1(m-2) = 0$

$\Rightarrow (m-2)[mx + (2m-1)] = 0$

$\Rightarrow m = 2$

By table

$\Rightarrow y = \underbrace{e^{2x}}_{u}$ is an integral of CF.

(iii) put $y = uv = e^{2x}.v$ in DE (i),

$$\frac{d^2v}{dx^2} + \left[-\frac{(2x+5)}{(x+2)} + \frac{2}{e^{2x}}.2e^{2x}\right]\frac{dv}{dx} = \frac{(x+1)e^x}{(x+2)e^{2x}}$$

$$\Rightarrow \frac{d^2v}{dx^2} + \frac{(2x+3)}{(x+2)}\frac{dv}{dx} = \frac{(x+1)e^{-x}}{(x+2)}$$

(iv) put $\frac{dv}{dx} = p$ in above DE

$$\Rightarrow \frac{dp}{dx} + \frac{(2x+3)}{(x+2)}p = \frac{(x+1)e^{-x}}{(x+2)} \quad \text{— (ii)}$$

[It is LDE in p] [first order]

$$I.f. = e^{\int \frac{(2x+3)}{(x+2)}dx}$$

$$= e^{\int\left[\frac{2x+4}{x+2} - \frac{1}{x+2}\right]dx}$$

$$= e^{2x - \log(x+2)} = \frac{e^{2x}}{(x+2)}$$

Sol^n of Eq.^{n} (ii) is

$$p \cdot \frac{e^{2x}}{x+2} = \int \frac{(x+1)}{(x+2)} e^{-x} \cdot \frac{e^{2x}}{(x+2)}\, dx + C$$ constant of integration

$$p \cdot \frac{e^{2x}}{(x+2)} = \int \frac{e^{x}}{(x+2)^2}(x+2-1)\, dx + C$$

$$= \int \left[\frac{e^{x}}{(x+2)} - \frac{e^{x}}{(x+2)^2}\right] dx + C$$

$$= \left[\frac{e^{x}}{(x+2)} + \cancel{\int \frac{e^{x}}{(x+2)^2} dx}\right] - \cancel{\int \frac{e^{x}}{(x+2)^2} dx} + C$$

$$= \frac{e^{x}}{(x+2)} + C$$

$$p = \frac{dv}{dx} = e^{-x} + C e^{-2x}(x+2)$$

on integration

$$v = -e^{-x} + C\left[\frac{(x+2)e^{-2x}}{-2} - \frac{e^{-2x}}{4}\right] + C_1$$

constant of integration

$$v = -e^{-x} + C\left[-\frac{(2x+5)}{4}\right]e^{-2x} + C_1.$$

Hence $y = uv = e^{2x}\left[-e^{-x} - \frac{C(2x+5)e^{-2x}}{4} + C_1\right]$

$$y = -e^{x} - \frac{C(2x+5)}{4} + C_1 e^{2x}$$ is required

CS.