(1)

Normal Form :- (By removal of First order derivative)

Working procedure:-

1.) Write the given DE in the standard form i.e.

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R \quad \text{-(i)}$$ (By making the coffecient of $\frac{d^2y}{dx^2}$ as unity)

2.) For removal of first derivative,

$$\boxed{u = e^{-\frac{1}{2}\int P dx}}$$

3.) Since $\boxed{y = uv} \xrightarrow{\text{is}}$ CS, it will reduce the DE (i) into normal form

$$\boxed{\frac{d^2v}{dx^2} + Iv = S}$$

where $I = Q - \frac{1}{4}P^2 - \frac{1}{2}\frac{dP}{dx}$, $S = \frac{R}{u} = Re^{\frac{1}{2}\int P dx}$

4.) There are two possible cases:-

(a) If $I =$ constant, then use $v = C.F. + P.I.$

(b) If $I = \left(\text{constant}/x^2\right)$, then the resulting Eq.^{n} is a homogeneous DE. Solve it by putting $z = \log x$. or $x = e^z$ & then $v = C.F. + P.I.$

(5) Determine v, so that complete solution is $\boxed{y = uv}$

② Ex:- $x^2\frac{d^2y}{dx^2} - 2(x^2+x)\frac{dy}{dx} + (x^2+2x+2)=0$

Solⁿ: → (i) $\frac{d^2y}{dx^2} + \left[-2\left(\frac{x^2}{x^2}+\frac{x}{x^2}\right)\right]\frac{dy}{dx} + \frac{(x^2+2x+2)}{x^2} = \frac{0}{x^2}$

$$\Rightarrow \frac{d^2y}{dx^2} + \underbrace{\left[-2\left(1+\frac{1}{x}\right)\right]}_{P}\frac{dy}{dx} + \underbrace{\left(1+\frac{2}{x}+\frac{2}{x^2}\right)}_{Q} = \underset{R}{\underset{\sim}{0}} \quad \text{—(i)}$$

(ii) To remove the first derivative.

$$u = e^{-\frac{1}{2}\int P dx} = e^{-\frac{1}{2}\int\left[-2\left(1+\frac{1}{x}\right)\right]dx} = e^{x+\log x}$$

$$u = e^x . e^{\log x} = e^x . x \quad \text{—(ii)}$$

(iii) put $\boxed{y = uv = e^x . xv}$ ← CS in Eqⁿ (i), ~~we have~~

$$\frac{d^2v}{dx^2} + Iv = S \quad \text{—(iii)}, \text{ where}$$

$$I = Q - \frac{1}{4}P^2 - \frac{1}{2}\frac{dP}{dx} = \left(1+\frac{2}{x}+\frac{2}{x^2}\right) - \frac{1}{\cancel{4}}\left[\cancel{4}\left(1+\frac{1}{x}\right)^2\right] - \frac{1}{\cancel{2}}\left[\frac{+\cancel{2}}{x^2}\right]$$

$$= 1 + \frac{2}{x} + \frac{2}{x^2} - 1 - \frac{2}{x} - \frac{1}{x^2} - \frac{1}{x^2} = 0$$

$$S = \frac{R}{u} = 0$$

Hence Eqⁿ (iii) becomes. $\frac{d^2v}{dx^2} = 0$

On Integration

$\Rightarrow \frac{dv}{dx} = c_1$ (constant of Integration)

$\Rightarrow \boxed{v = c_1 x + c_2}$ —(iv) (where c_2 is constant of Integration

from (ii) & (iv)

$\boxed{y = uv = e^x . x\,[c_1 x + c_2]}$ is CS of (i)

③ Ex:- $\frac{d^2y}{dx^2} + \underbrace{\frac{1}{x^{1/3}}} \frac{dy}{dx} + \underbrace{\left(\frac{1}{4x^{2/3}} - \frac{1}{6x^{4/3}} - \frac{6}{x^2}\right)} y = \underbrace{0}$ —(i)

Solⁿ:→ (i) $P = \frac{1}{x^{1/3}}$, $Q = \left(\frac{1}{4x^{2/3}} - \frac{1}{6x^{4/3}} - \frac{6}{x^2}\right)$, $R = 0$

(ii) For removal of first derivative,

$$u = e^{-\frac{1}{2}\int P dx} = e^{-\frac{1}{2}\int \frac{1}{x^{1/3}} dx} = e^{-\frac{3}{4}x^{2/3}} \quad \text{—(ii)}$$

(iii) put $\boxed{y = uv = e^{-\frac{3}{4}x^{2/3}} \cdot v}$ —(iii) in DE (i) ← CS

$$\frac{d^2v}{dx^2} + Iv = S \quad \text{—(iv) where}$$

$$I = Q - \frac{1}{4}P^2 - \frac{1}{2}\frac{dP}{dx}, \quad S = \frac{R}{u}$$

$$\Rightarrow I = \left(\cancel{\frac{1}{4x^{2/3}}} - \cancel{\frac{1}{6x^{4/3}}} - \frac{6}{x^2}\right) - \frac{1}{4}\left(\cancel{\frac{1}{x^{2/3}}}\right) - \frac{1}{2}\left(\cancel{-\frac{1}{3}\frac{1}{x^{4/3}}}\right); \; S = 0$$

$\Rightarrow I = -\frac{6}{x^2}$, $S = 0$ Hence Eqⁿ (iv) becomes

$$\frac{d^2v}{dx^2} + \left(-\frac{6}{x^2}\right)v = 0 \quad \text{(Step (iv), case (b) is applicable)}$$

$$\Rightarrow x^2\frac{d^2v}{dx^2} - 6v = 0 \quad \text{—(v) (Homogeneous DE)}$$

put $z = \log x$ or $x = e^z$

$[D(D-1)v - 6v] = 0 \Rightarrow [\underbrace{D(D-1) - 6}_{f(D)}]\, v = 0$ $\left[\text{where } D = \frac{d}{dz}\right]$

For C.F. :→ Auxiliary eqⁿ is $f(m) = 0$

$$\Rightarrow m(m-1) - 6 = 0$$
$$\Rightarrow m^2 - 3m + 2m - 6 = 0$$
$$\Rightarrow (m+2)(m-3) = 0$$
$$\Rightarrow m = -2, 3$$

(4)

$$C.F. = c_1 e^{-2z} + c_2 e^{3z}$$
$$= c_1 x^{-2} + c_2 x^3 \quad (\text{as } x = e^z)$$

$$P.I. = \frac{1}{f(D)} \cdot 0 = 0$$

$$\Rightarrow \quad v = C.F. + P.I. = c_1 x^{-2} + c_2 x^3 \quad \text{—— (vi)}$$

Hence by (ii) & (vi),

$$y = uv = e^{-\frac{3}{4}x^{2/3}} (c_1 x^{-2} + c_2 x^3) \text{ is CS of (i).}$$

Ex:→ $$\frac{d^2y}{dx^2} - 2\tan x \frac{dy}{dx} + 5y = e^x \sec x \quad \text{—— (i)}$$

Solⁿ→ (i) $P = -2\tan x$, $Q = 5$, $R = e^x \sec x$

(ii) To remove the first derivative.

$$u = e^{-\frac{1}{2}\int P\,dx} = e^{-\frac{1}{2}\int(-2\tan x)\,dx} = e^{\log \sec x} = \sec x \text{ —— (ii)}$$

(iii) put $\boxed{y = uv = (\sec x)\, v} \leftarrow CS$ —— (iii) in DE (i),

$$\frac{d^2v}{dx^2} + Iv = S \quad \text{—— (iv) where}$$

$$I = 5 - \frac{1}{4}(4\tan^2 x) - \frac{1}{2}(-2\sec^2 x), \quad S = \frac{R}{u} = \frac{e^x \cancel{\sec x}}{\cancel{\sec x}}$$

$$, S = e^x$$

$$I = 5 + 1 = 6$$

Hence (iv) becomes.

$$\frac{d^2v}{dx^2} + 6v = e^x \Rightarrow \underbrace{(D^2+6)}_{f(D)} v = e^x$$

For C.F. :– A. Eqⁿ is $f(m) = 0 \Rightarrow m^2 + 6 = 0$
$$\Rightarrow m = \pm\sqrt{6}\, i$$

$$C.F. = e^{0.x}(c_1 \cos\sqrt{6}x + c_2 \sin\sqrt{6}x)$$

$$P.I. = \frac{1}{f(D)} \cdot e^x = \frac{1}{D^2+6} \cdot e^x$$

$\left[P.I = \frac{1}{f(D)} e^{ax} \quad \text{then } D \to a\right]$

$$= \frac{e^x}{1+6} = \frac{e^x}{7}$$

Therefore v = C.F. + P.I.

$$= (c_1 \cos\sqrt{6}x + c_2 \sin\sqrt{6}x) + \frac{e^x}{7} \quad \text{---(V)}$$

$y = uv = \sec x\left[(c_1 \cos\sqrt{6}x + c_2 \sin\sqrt{6}x) + \frac{e^x}{7}\right]$ is CS.

Ex:- $\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + (4x^2-1)y = -3e^{x^2}\sin 2x$ —(i)

Sol^n :- $P = -4x$, $Q = 4x^2-1$, $R = -3e^{x^2}\sin 2x$

To remove the first derivative.

$$u = e^{-\frac{1}{2}\int P\,dx} = e^{-\frac{1}{2}\int(-4x)dx} = e^{x^2} \quad \text{—(ii)}$$

$I = (4x^2-1) - \frac{1}{4}(16x^2) - \frac{1}{2}(-4) = 1$, $S = \frac{R}{u} = -3\sin 2x$

put $\boxed{y = uv = e^{x^2}.v}$ —(iii) in DE (i), we have $\to$ CS

$\frac{d^2v}{dx^2} + (1)v = -3\sin 2x$ —(iv) $\Rightarrow (D^2+1)v = -3\sin 2x$

C.F. :→ A. Eq.^{n} is $m^2+1=0 \Rightarrow m = \pm i$

C.F. $= c_1\cos x + c_2 \sin x$

P.I. $= \frac{1}{(D^2+1)}(-3\sin 2x)$ $\quad [D^2 \to -a^2 \text{ for } \sin ax]$ i.e. $D^2 \to -2^2$

$$= \frac{-3\sin 2x}{(-4+1)} = \sin 2x$$

v = C.F. + P.I. $= c_1\cos x + c_2\sin x + \sin 2x$ —(V)

Hence $y = uv = e^{x^2}[c_1\cos x + c_2\sin x + \sin 2x]$ is CS of (i) [by (ii) & (V)]