(1)

Change of independent variable:→

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R \quad \text{—(i)}$$ Here P, Q, R are funⁿ of x.

$x \longrightarrow z.$

$$P_1 = \frac{\frac{d^2z}{dx^2} + P\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2}, \quad Q_1 = \frac{Q}{\left(\frac{dz}{dx}\right)^2}, \quad R_1 = \frac{R}{\left(\frac{dz}{dx}\right)^2} \quad \text{—(*)}$$

→ selection of the variable z. is done by.

$Q_1 =$ constant.

case-I If $P_1 = 0$ then

(a) $Q_1 =$ const. then solve it by method of LDE with constant coffecients.

(b) If $Q_1 = \frac{\text{const}}{z^2}$ then by homogeneous DE.

case-II If $Q_1 = a^2$

$$\frac{d^2y}{dz^2} + P_1\frac{dy}{dz} + Q_1 y = R_1$$

Ex:→ $$\frac{d^2y}{dx^2} + \frac{2}{x}\frac{dy}{dx} + \frac{a^2}{x^4}y = 0 \quad \text{—(i)}$$

Solⁿ:→ $P = \frac{2}{x}$, $Q = \frac{a^2}{x^4}$, $R = 0$

for changing x to z

(2)

$$Q_1 = \frac{Q}{\left(\frac{dz}{dx}\right)^2} = \text{constant} = a^2 \quad \text{— (ii)}$$

$$\Rightarrow \frac{a^2/x^4}{\left(\frac{dz}{dx}\right)^2} = 1 \text{ or } a^2$$

$$\Rightarrow \frac{\cancel{a^2}/x^4}{\left(\frac{dz}{dx}\right)^2} = \cancel{a^2} \text{ (let)}$$

$$\Rightarrow \frac{1}{x^4} = \left(\frac{dz}{dx}\right)^2 \Rightarrow \frac{1}{x^2} = \frac{dz}{dx}$$

On integration

$$\Rightarrow -\frac{1}{x} = z$$

$$\Rightarrow \boxed{z = -\frac{1}{x}} \quad \text{— (iii)}$$

$$P_1 = \frac{\frac{d^2z}{dx^2} + P\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2}, \quad R_1 = \frac{R}{\left(\frac{dz}{dx}\right)^2}$$

$$P_1 = \frac{-\frac{2}{x^3} + \left(\frac{2}{x}\right)\left(\frac{1}{x^2}\right)}{\left(\frac{1}{x^2}\right)^2} = 0, \quad R_1 = \frac{0}{1/x^2} = 0 \quad \text{— (iv)}$$

Hence changing indeper. var. x to z, Eq.^{n} (i) becomes.

$$\frac{d^2y}{dz^2} + P_1\frac{dy}{dz} + Q_1 y = R_1$$

$$\Rightarrow \frac{d^2y}{dz^2} + 0 \cdot \frac{dy}{dz} + a^2 y = 0 \quad \text{(by (ii) \& (iv))}$$

(8)

$$\Rightarrow \frac{d^2y}{dz^2} + a^2y = 0 \Rightarrow \underbrace{(D^2+a^2)}_{f(D)}y = 0$$

where $D = \frac{d}{dz}$

$y = C.F. + P.I.$

C.F. :→ A. Eq^n is $f(m) = 0 \Rightarrow m^2 + a^2 = 0$

$\Rightarrow m^2 = -a^2$

$\Rightarrow m = 0 \pm ai$

$$C.F. = e^{0.z}[c_1 \cos az + c_2 \sin az]$$

$$P.I. = \frac{1}{f(D)} \cdot 0 = 0$$

$y = C.F. + P.I.$

$= c_1 \cos az + c_2 \sin az + 0$

$y = c_1 \cos\left(-\frac{a}{x}\right) + c_2 \sin\left(-\frac{a}{x}\right)$ Ans (by (iii))

Ex:→ $x\frac{d^2y}{dx^2} - \frac{dy}{dx} - 4x^3 \cdot y = 4x^3 \cdot \sin x^2$

Sol^n:→ $\frac{d^2y}{dx^2} + \underbrace{\left[-\frac{1}{x}\right]}_{P}\frac{dy}{dx} + \underbrace{[-4x^2]}_{Q} \cdot y = \underbrace{4x^2 \sin x^2}_{R}$ —(i)

Changing independent variable. x to z

$Q_1 =$ constant $= (-1)$ —(ii)

$$\Rightarrow \frac{Q}{\left(\frac{dz}{dx}\right)^2} = \text{const}$$

(9)

$$\Rightarrow \frac{(\cancel{-}4x^2)}{\left(\frac{dz}{dx}\right)^2} = (\cancel{-1}) \Rightarrow (2x)^2 = \left(\frac{dz}{dx}\right)^2$$

$$\Rightarrow 2x = \frac{dz}{dx}$$

On integration

$$\Rightarrow \cancel{2}\cdot\frac{x^2}{\cancel{2}} = z \quad \text{—— (iii)}$$

$$P_1 = \frac{\frac{d^2z}{dx^2} + P\left(\frac{dz}{dx}\right)}{\left(\frac{dz}{dx}\right)^2}, \quad R_1 = \frac{R}{\left(\frac{dz}{dx}\right)^2}$$

$$= \frac{2 + \left(\frac{-1}{x}\right)(2x)}{(2x)^2}, \quad R_1 = \frac{4x^2 \sin x^2}{(2x)^2}$$

$$P_1 = 0, \quad R_1 = \sin x^2 = \sin z \text{ (by (iii))}$$

Hence our DE becomes

$$\frac{d^2y}{dz^2} + 0\cdot\frac{dy}{dx} + (-1)\cdot y = \sin z \quad \text{—— (iv) where } D = \frac{d}{dz}$$

$$\Rightarrow \underbrace{[D^2 - 1]}_{f(D)} y = \sin z$$

C.F. :→ A. Eqⁿ is $f(m) = 0$

$$m^2 - 1 = 0 \Rightarrow m^2 = 1$$

$$\Rightarrow m = \pm 1$$

$$C.F. = C_1 e^{1\cdot z} + C_2 e^{-1\cdot z}$$

$$P.I. = \frac{1}{f(D)}\cdot \sin z$$

$$= \frac{1}{D^2 - 1} \sin z$$

$$(D^2 \to -a^2 \text{ in } \sin az)$$

$$D^2 \to -1^2$$

$$= \frac{1}{-1-1} \sin z$$

$$= -\frac{1}{2} \sin z$$

$y = C.F. + P.I.$

$= c_1 e^{z} + c_2 e^{-z} - \frac{1}{2} \sin z$

$y = c_1 e^{x^2} + c_2 e^{-x^2} - \frac{1}{2} \sin x^2$ is our CS

(by (iii))