(10)

$y = C.F. + P.I.$

$= c_1 e^{z} + c_2 e^{-z} - \frac{1}{2} \sin z$

$y = c_1 e^{x^2} + c_2 e^{-x^2} - \frac{1}{2} \sin x^2$ is our CS

(by (iii))

Ex:→ $x^6 \frac{d^2y}{dx^2} + 3x^5 \frac{dy}{dx} + a^2 y = \frac{1}{x^2}$.

Sol^n:→ $\frac{d^2y}{dx^2} + \underbrace{\frac{3}{x}}_{P} \frac{dy}{dx} + \underbrace{\frac{a^2}{x^6}}_{Q} y = \underbrace{\frac{1}{x^8}}_{R}$ —(i)

change of Independent var. $x \to z$

$Q_1 =$ constant $= 1$ —(ii)

$\frac{Q}{\left(\frac{dz}{dx}\right)^2} = 1 \Rightarrow \frac{a^2/x^6}{\left(\frac{dz}{dx}\right)^2} = 1$

$\Rightarrow \frac{a^2}{x^6} = \left(\frac{dz}{dx}\right)^2 \Rightarrow \frac{a}{x^3} = \frac{dz}{dx}$ —(iii)

On Integration

$\Rightarrow \boxed{a\left(-\frac{1}{2x^2}\right) = z}$ —(iv)

$P_1 = \frac{\frac{d^2z}{dx^2} + P\left(\frac{dz}{dx}\right)}{\left(\frac{dz}{dx}\right)^2}$, $R_1 = \frac{R}{\left(\frac{dz}{dx}\right)^2}$

(11.)

$$P_1 = \frac{-\frac{3a}{x^4} + \left(\frac{3}{x}\right)\left(\frac{a}{x^3}\right)}{\left(\frac{a}{x^3}\right)^2} \quad , \quad R_1 = \frac{1/x^8}{\left(a/x^3\right)^2}$$

$$P_1 = 0 \quad , \quad R_1 = \frac{1}{a^2x^2} = \frac{1}{a^2}\left(-\frac{2z}{a}\right)$$

$$\frac{d^2y}{dz^2} + P_1 \cdot \frac{dy}{dz} + Q_1 y = R_1 \Rightarrow \frac{d^2y}{dz^2} + y = \frac{-2z}{a^3}$$

$$\underbrace{(D^2+1)}_{f(D)} y = -\frac{2z}{a^3} \quad \text{where } D = \frac{d}{dz}$$

C.F :→ A. Eq.^{n} is $f(m) = 0 \Rightarrow m^2 + 1 = 0$

$$\Rightarrow m^2 = -1$$

$$\Rightarrow m = 0 \pm i$$

$$\text{C.F.} = e^{0 \cdot z}\left[C_1 \cos z + C_2 \sin z\right]$$

$$\text{P.I.} = \frac{1}{f(D)} \cdot \left(-\frac{2z}{a^3}\right) = -\frac{2}{a^3} \frac{1}{(1+D^2)} \cdot z$$

$$= -\frac{2}{a^3}(1+D^2)^{-1} \cdot z^1$$

$$= -\frac{2}{a^3}(1 - D^2 + \cdots\cdots). z^1$$

$$= -\frac{2}{a^3}(z - 0 + 0 + 0 \cdots)$$

$$= -\frac{2z}{a^3}$$

$$y = C.F. + P.I.$$

$$= c_1 \cos z + c_2 \sin z - \frac{2z}{a^3}$$

$$= c_1 \cos\left(\frac{a}{2x^2}\right) + c_2 \sin\left(\frac{a}{2x^2}\right) - \frac{\cancel{2}}{a^3}\left(\frac{a}{\cancel{2}x^2}\right)$$

Ex:→ $\cos x \frac{d^2y}{dx^2} + \sin x \frac{dy}{dx} - 2\cos^3 x\, y = 2\cos^5 x$

Sol^n:→ $\frac{d^2y}{dx^2} + \underbrace{\tan x}_{P} \frac{dy}{dx} + \underbrace{(-2\cos^2 x)}_{Q} y = \underbrace{2\cos^4 x}_{R}$ —(i)

$x \to z$

$Q_1 = \text{Constant} \Rightarrow \frac{Q}{\left(\frac{dz}{dx}\right)^2} = \text{constant}$

$\Rightarrow \frac{-2\cos^2 x}{\left(\frac{dz}{dx}\right)^2} = (-2)$ (let) $\Rightarrow \cos^2 x = \left(\frac{dz}{dx}\right)^2$

$\Rightarrow \cos x = \left(\frac{dz}{dx}\right)$ —(ii) $\Rightarrow \boxed{\sin x = z}$ —(iii)

$P_1 = \frac{\frac{d^2z}{dx^2} + P\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2}$, $R_1 = \frac{R}{\left(\frac{dz}{dx}\right)^2}$

$P_1 = \frac{-\sin x + (\tan x)\cos x}{\cos^2 x}$, $R_1 = \frac{2\cos^4 x}{\cos^2 x}$

$P_1 = 0$, $R_1 = 2\cos^2 x$

$= 2(1 - \sin^2 x)$

$= 2(1 - z^2)$

$$\frac{d^2y}{dz^2} + (-2)y = 2(1-z^2)$$

$$\Rightarrow [D^2 + (-2)]y = 2(1-z^2) \text{ where } D = \frac{d}{dz}$$

C.F. :→ $f(m) = 0 \Rightarrow m^2 - 2 = 0 \Rightarrow m = \pm\sqrt{2}$

$$\text{C.F.} = c_1 e^{\sqrt{2}z} + c_2 e^{-\sqrt{2}z}$$

$$\text{P.I.} = \frac{1}{(D^2-2)} 2(1-z^2) = \frac{2}{(-2)}\left(1 - \frac{D^2}{2}\right)^{-1}(1-z^2)$$

$$= (-1)\left[1 + \frac{D^2}{2} + \cdots\right](1-z^2)$$

$$= (-1)\left[1 - z^2 + \frac{1}{2}(0-2) + 0\right]$$

$$= (-1)(-z^2) = z^2$$

$$y = \text{C.F.} + \text{P.I.}$$

$$y = c_1 e^{\sqrt{2}z} + c_2 e^{-\sqrt{2}z} + z^2$$

$$y = c_1 e^{\sqrt{2}\sin x} + c_2 e^{-\sqrt{2}\sin x} + \sin^2 x \quad \text{Ans}$$

Ex :→ $\dfrac{d^2y}{dx^2} + \underbrace{(3\sin x - \cot x)}_{P}\dfrac{dy}{dx} + \underbrace{2\sin^2 x}_{Q} \cdot y = \underbrace{e^{-\cos x}\sin^2 x}_{R}$ —— (i)

Sol^n :→ take Q_1 = constant = 2

$$\Rightarrow \frac{Q}{\left(\frac{dz}{dx}\right)^2} = 2 \Rightarrow \cancel{2}\sin^2 x = \cancel{2}\left(\frac{dz}{dx}\right)^2$$

$$\Rightarrow \sin x = \frac{dz}{dx} \text{ —— (ii)}$$

$$\boxed{z = -\cos x} \text{ --- (iii)}$$

$$P_1 = \frac{\frac{d^2z}{dx^2} + P\left(\frac{dz}{dx}\right)}{\left(\frac{dz}{dx}\right)^2}, \quad R_1 = \frac{R}{\left(\frac{dz}{dx}\right)^2}$$

$$P_1 = \frac{\cancel{\cos x} + (3\sin x - \cancel{\cot x})\sin x}{\sin^2 x}, \quad R_1 = \frac{e^{-\cos x}\cancel{\sin^2 x}}{\cancel{\sin^2 x}}$$

$$P_1 = 3, \quad R_1 = e^{-\cos x} = e^{z}$$

so our transformed DE is

$$\frac{d^2y}{dz^2} + 3\frac{dy}{dz} + 2y = e^{z} \Rightarrow \underbrace{(D^2+3D+2)}_{f(D)}y = e^{z}$$

C.F :→ A. Eq^n is $f(m)=0 \Rightarrow m^2+3m+2=0$

$\Rightarrow (m+1)(m+2) = 0 \Rightarrow m = -1, -2$

$$C.F. = c_1 e^{-z} + c_2 e^{-2z}$$

$$P.I. = \frac{1}{(D+1)(D+2)} e^{1.z} \quad \left[\begin{matrix} D \to a \\ D \to 1 \end{matrix} \quad e^{az}\right]$$

$$= \frac{e^{z}}{(1+1)(1+2)} = \frac{e^{z}}{6}$$

$$y = C.F. + P.I.$$

$$= c_1 e^{-z} + c_2 e^{-2z} + \frac{e^{z}}{6}$$

$$y = c_1 e^{\cos x} + c_2 e^{2\cos x} + \frac{1}{6} e^{-\cos x}$$