Method of variation of parameter

(i) $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$ — (i)

(ii) Take $R = 0$

$\Rightarrow \frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = 0$

let C.F. of Eq.^{n} (i) be

C.F. $= C_1 u + C_2 v$

(iii) let G.S. of (i) be

$y = Au + Bv$ — (ii)

are fun.^{n} of x

(iv) choose A & B s.t.

$u\frac{dA}{dx} + v\frac{dB}{dx} = 0$ — (iii)

$\frac{du}{dx}\frac{dA}{dx} + \frac{dv}{dx}\frac{dB}{dx} = R$ — (iv)

(v) Solve (iii) & (iv), we get $\frac{dA}{dx}, \frac{dB}{dx}$,

On integration, we get A & B.

put value of A & B in (ii) will give us G.S.

Ex:→ Solve by method of variation of parameter $\frac{d^2y}{dx^2} + a^2 y = \sec ax$

Soln:→ $\underbrace{(D^2+a^2)}_{f(D)} y = \sec ax \quad \text{— (i)}$

Taking $R=0 \Rightarrow (D^2+a^2)\, y = 0$

C.F :→ A. Eq.n is $f(m)=0 \Rightarrow m^2+a^2=0$

$\Rightarrow m^2 = -a^2$

$\Rightarrow m = 0 \pm ai$

$$C.F. = e^{0.x}(c_1 \cos ax + c_2 \sin ax)$$

let the G.S. of eq.n (i) be

$$y = A \underbrace{\cos ax}_{u} + B \underbrace{\sin ax}_{v} \quad \text{— (ii)}$$

where A & B are fun.n of x s.t.

$$\cos ax \frac{dA}{dx} + \sin ax \frac{dB}{dx} = 0 \quad \text{— (iii)}$$

$$-a \sin ax \frac{dA}{dx} + a \cos ax \frac{dB}{dx} = \sec ax \quad \text{— (iv)}$$

(iii) X $a \sin ax$ + (iv) X $\cos ax$

$$0 + a \sin^2 ax \frac{dB}{dx} + a \cos^2 ax \frac{dB}{dx} = \sec ax \cos ax$$

$$a(\sin^2 ax + \cos^2 ax)\frac{dB}{dx} = 1$$

$$\Rightarrow a\frac{dB}{dx} = 1 \Rightarrow \frac{dB}{dx} = \frac{1}{a} \quad \text{— (v)}$$

On integration

$$\boxed{B = \frac{x}{a} + c_1'}$$ where c_1' is constant of integration.

from Eq.^{n} (iii) $\frac{dA}{dx} = -\tan ax \frac{dB}{dx}$

$$\frac{dA}{dx} = -\frac{1}{a}\tan ax$$

On integration

$$\boxed{A = \frac{1}{a^2}\log \cos ax + c_2'}$$

put value of A & B in (ii), G S is

$$y = \left\{\frac{1}{a^2}\log \cos ax + c_2'\right\}\cos ax + \left(\frac{x}{a} + c_1'\right)\sin ax$$

Ex:→ solve by method of var. of parameter.

$$\frac{d^2y}{dx^2} - y = \frac{2}{1+e^x}$$

sol^n:→ $\underbrace{(D^2-1)}_{f(D)} y = \frac{2}{(1+e^x)}$ ——— (i)

C. F. :→ A. Eq.^{n} be $f(m) = 0$

$$m^2 - 1 = 0$$

$$\Rightarrow m = \pm 1$$

$$\therefore C.F. = c_1 e^x + c_2 e^{-x}$$

let GS of (i) be

$$y = A\underbrace{e^{x}}_{u} + B\underbrace{e^{-x}}_{v} \quad \text{—(ii)}$$ where A & B are

fun.^{n} of x s.t.

$$e^{x}\frac{dA}{dx} + e^{-x}\frac{dB}{dx} = 0 \quad \text{—(iii)}$$

$$e^{x}.\frac{dA}{dx} - e^{-x}\frac{dB}{dx} = \frac{2}{(1+e^{x})} \quad \text{—(iv)}$$

(iii) + (iv) $\Rightarrow \cancel{2}e^{x}\frac{dA}{dx} = \frac{\cancel{2}}{(1+e^{x})}$

$$\Rightarrow \frac{dA}{dx} = \frac{e^{-x}}{(1+e^{x})} = \frac{1}{e^{x}(1+e^{x})}$$

$$\Rightarrow \frac{dA}{dx} = \frac{1}{e^{x}} - \frac{1}{1+e^{x}} = e^{-x} - \frac{e^{-x}}{(e^{-x}+1)}$$

on integration

$$\Rightarrow A = -e^{-x} - \int\frac{e^{-x}}{(1+e^{-x})}dx + k_1$$

let $(1+e^{-x}) = t$

$-e^{-x}dx = dt$

$$\boxed{A = -e^{-x} + \log(1+e^{-x}) + k_1}$$

from eq.^{n} (iii) $\Rightarrow e^{-x}\frac{dB}{dx} = e^{x}\frac{dA}{dx} = e^{x}\left[\frac{e^{-x}}{(1+e^{x})}\right]$

$$\Rightarrow \frac{dB}{dx} = \frac{e^{x}}{1+e^{x}}$$ on integration

$$\Rightarrow \boxed{B = \log(1+e^{x}) + k_2}$$

from eq.^{n} (ii)

$$\Rightarrow y = [-e^{-x} + \log(1+e^{-x}) + k_1]e^{x} + [\log(1+e^{x}) + k_2]e^{-x}$$ is G.S.