Unit-I Exact linear differential Equation

Linear D.E of n^{th} order

$$P_0 \frac{d^n y}{dx^n} + P_1 \frac{d^{n-1}y}{dx^{n-1}} + \dots + P_n y = Q \quad \text{— (i)}$$

condition of Exactness for above Eq.^{n} is

$$+P_n - P'_{n-1} + P''_{n-2} - P'''_{n-3} + \dots = 0 \quad \text{— (ii)}$$

If Eq.^{n} (i) is Exact, than its first integral is given by.

$$P_0 \frac{d^{n-1}y}{dx^{n-1}} + (P_1 - P_0') \frac{d^{n-2}y}{dx^{n-2}} + (P_2 - P_1' + P_0'') \frac{d^{n-3}y}{dx^{n-3}} + \dots$$

$$+ \dots + (P_{n-1} - P'_{n-2} + P''_{n-3} \dots) y = \int Q\,dx + c$$

— (iii)

Repeat this process, to get final integral of (i)

Ex:-1 solve.

$$\underbrace{(1+x+x^2)}_{P_0} \frac{d^3y}{dx^3} + \underbrace{(3+6x)}_{P_1} \frac{d^2y}{dx^2} + \underset{P_2}{6} \frac{dy}{dx} + \underset{P_3}{0} \cdot y = 0 \quad \text{— (i)}$$

Sol^n:- condition of exactness.

$$P_3 - P_2' + P_1'' - P_0'' = 0 - 0 + 0 - 0 = 0$$

Hence condition of exactness is satisfied

So first integral of (i) is given by.

$$P_0 \frac{d^2y}{dx^2} + (P_1 - P_0') \frac{dy}{dx} + (P_2 - P_1' + P_0'')y = C_1$$

or $$(1+x+x^2)\frac{d^2y}{dx^2} + \{(3+6x) - (1+2x)\}\frac{dy}{dx} + (6-6+2)y = C_1$$

or $$\underbrace{(1+x+x^2)}_{P_0}\frac{d^2y}{dx^2} + \underbrace{(2+4x)}_{P_1}\frac{dy}{dx} + \underbrace{2}_{P_2}y = \underbrace{C_1}_{Q} \quad \text{--- (ii)}$$

Again $P_2 - P_1' + P_0'' = 2-4+2 = 0$

So Eq.^{n} (ii) is also a exact DE, Hence its first integral is

$$P_0 \frac{dy}{dx} + (P_1 - P_0')y = \int C_1\, dx + C_2$$

or $$(1+x+x^2)\frac{dy}{dx} + \{(2+4x) - (1+2x)\}y = C_1x + C_2$$

or $$\underbrace{(1+x+x^2)}_{P_0}\frac{dy}{dx} + \underbrace{(1+2x)}_{P_1}y = \underbrace{C_1x + C_2}_{Q} \quad \text{--- (iii)}$$

$$P_1 - P_0' = (1+2x) - (1+2x) = 0$$

$\Rightarrow$ Eq.^{n} (iii) is also a exact DE, Hence its first integral is given by.

$$P_0 y = \int Q\, dx + C_3$$

$$\Rightarrow (1+x+x^2)y = \int (C_1x + C_2)\,dx + C_3$$

$$\Rightarrow (1+x+x^2)y = C_1\frac{x^2}{2} + C_2x + C_3 \quad \text{Ans}$$

Ex:→ Solve. ③

$$\underbrace{(2x^2+3x)}_{P_0}\frac{d^2y}{dx^2}+\underbrace{(6x+3)}_{P_1}\frac{dy}{dx}+\underbrace{2}_{P_2}y=\underbrace{(x+1)e^x}_{Q} \quad \text{—(i)}$$

Solⁿ:- since $P_2-P_1'+P_0''=2-6+4=0$

Hence condition of Exactness is satisfied.

so first integral of eqⁿ (i) is given by.

$$P_0\frac{dy}{dx}+(P_1-P_0')y=\int Q\,dx+c$$

$$\Rightarrow (2x^2+3x)\frac{dy}{dx}+\{(6x+\cancel{3})-(4x+\cancel{3})\}y=\int(x+1)e^x dx+c$$

$$\Rightarrow (2x^2+3x)\frac{dy}{dx}+2xy=\{(x+1)e^x-e^x\}+c$$

$$\Rightarrow \underbrace{(2x^2+3x)}_{P_0}\frac{dy}{dx}+\underbrace{2x}_{P_1}y=\underbrace{xe^x+c}_{Q} \quad \text{—(ii)}$$

$P_1-P_0'\neq 0\Rightarrow$ condition of exactness is not satisfied.

⇒ Eqⁿ (ii) is not Exact.

$$\text{Eq}^n\text{ (ii)}\Rightarrow \frac{dy}{dx}+\frac{2\cancel{x}}{\cancel{x}(2x+3)}y=\frac{\cancel{x}e^x}{\cancel{x}(2x+3)}+\frac{c}{x(2x+3)} \quad \text{—(iii)}$$

$$I.f.=e^{\int\frac{2}{2x+3}dx}=e^{\cancel{2}\left[\frac{1}{\cancel{2}}\log(2x+3)\right]}=(2x+3)$$

Hence solⁿ. of eqⁿ (iii) is given by.

$$y.(2x+3)=\int\left[\frac{e^x}{\cancel{(2x+3)}}+\frac{c}{x\,\cancel{(2x+3)}}\right]\cancel{(2x+3)}dx+c_1$$

$$y\cdot(2x+3) = \int\left(e^x + \frac{c}{x}\right)dx + C_1$$

$$y\cdot(2x+3) = e^x + c\log x + C_1 \quad \text{Ans.}$$

Ex :- 3) $\underbrace{1}_{P_0}\cdot\frac{d^3y}{dx^3} + \underbrace{\cos x}_{P_1}\frac{d^2y}{dx^2} \underbrace{- 2\sin x}_{P_2}\frac{dy}{dx} \underbrace{- \cos x}_{P_3}\, y = \underbrace{\sin 2x}_{Q}$

Solⁿ:- $P_0 = 1,\ P_1 = \cos x,\ P_2 = -2\sin x,\ P_3 = -\cos x$

$Q = \sin 2x$

$$P_3 - P_2' + P_1'' - P_0''' = -\cos x - (-2\cos x) - \cos x - 0 = 0$$

$\Rightarrow$ given DE is Exact DE, Hence its first integral is given by.

$$P_0\frac{d^2y}{dx^2} + (P_1 - P_0')\frac{dy}{dx} + (P_2 - P_1' + P_0'')y = \int Q\,dx + c$$

$$\Rightarrow 1\cdot\frac{d^2y}{dx^2} + (\cos x - 0)\frac{dy}{dx} + (-2\sin x + \sin x + 0)y = \int \sin 2x\,dx + c$$

$$\Rightarrow \underbrace{1}_{P_0}\cdot\frac{d^2y}{dx^2} + \underbrace{\cos x}_{P_1}\frac{dy}{dx} + \underbrace{(-\sin x)}_{P_2}y = \underbrace{-\frac{1}{2}\cos 2x + C_1}_{Q} \quad \text{(i)}$$

$$P_2 - P_1' + P_0'' = -\sin x + \sin x + 0 = 0$$

$\Rightarrow$ Eqⁿ (i) is Exact DE, hence its first integral is given by.

$$P_0\frac{dy}{dx} + (P_1 - P_0')y = \int Q\,dx + C_2$$

$$\frac{dy}{dx} + \cos x\cdot y = \int\left[-\frac{1}{2}\cos 2x + C_1\right]dx + C_2$$

$$\frac{dy}{dx} + \cos x \cdot y = -\frac{1}{4}\sin 2x + C_1 x + C_2 \quad \text{—(ii)}$$

since eq^n (ii) is not Exact DE but it is LDE of first order

$$I.f. = e^{\int \cos x \, dx} = e^{\sin x}$$

Hence sol^n of eq^n (ii) is given by

$$y(e^{\sin x}) = \int \left(-\frac{1}{4}\sin 2x + C_1 x + C_2\right) e^{\sin x} dx + C_3$$

$$y \cdot e^{\sin x} = \int (C_1 x + C_2) e^{\sin x} dx - \frac{1}{2}\underbrace{\int \sin x \cos x \, e^{\sin x} dx}_{*} + C_3$$

$\{$ In $(*)$ let $\sin x = t \Rightarrow \cos x \, dx = dt \}$

$$y \cdot e^{\sin x} = \int (C_1 x + C_2) e^{\sin x} dx - \frac{1}{2}(t e^t - e^t) + C_3$$

$$= \int [C_1 x + C_2] e^{\sin x} dx - \frac{1}{2}(\sin x \, e^{\sin x} - e^{\sin x}) + C_3$$

Ans