Integrating factor :-

Sometimes the DE is not given in exact form but it can be reduced to exact DE by multiplying it with integrating factor.

(i) If the coefficient $P_0, P_1, \ldots$ are Polynomials than I.f. is of the form x^m

(ii) If the coefficient $P_0, P_1, \ldots$ are Tri. functi[on] than I.f. is of also of the form Tri. fun.^{n}.

Ex:→ Solve $x^3 \frac{d^2y}{dx^2} + 2x\frac{dy}{dx} - 2y = 1$

Sol?- Since the given DE is not exact.

(all the coeff^n P_0, P_1, P_2 are polynomials in x)

Therefore multiplying it with integrating factor x^m

$$\underbrace{x^{m+3}}_{P_0}\frac{d^2y}{dx^2} + \underbrace{2x^{m+1}}_{P_1}\frac{dy}{dx} - \underbrace{2x^m}_{P_2}y = \underbrace{x^m}_{Q} \quad \text{—(i)}$$

since Eq^n (i) is exact DE therefore

$$P_2 - P_1' + P_0'' = 0$$

$$\Rightarrow -2x^m - [2(m+1)x^m] + [(m+3)(m+2)x^{m+1}] = 0$$

$$\Rightarrow x^m[-2-2(m+1)] + [(m+3)(m+2)x^{m+1}] = 0$$

$$\Rightarrow x^m[-2(m+2)] + (m+3)(m+2)x^{m+1} = 0$$

$$\Rightarrow (m+2)[-2x^m + (m+3)x^{m+1}] = 0$$

$\{ a\cdot b = 0 \Rightarrow a=0 \text{ or } b=0 \}$

$$\Rightarrow m+2=0 \Rightarrow m=-2$$

put value of m in Eq^n (i)

$$x\frac{d^2y}{dx^2} + \frac{2}{x}\frac{dy}{dx} - \frac{2}{x^2}y = \frac{1}{x^2} \quad \text{—(ii)}$$

Eq.^{n} (ii) is Exact DE; hence its first integral is

$$P_0\frac{dy}{dx} + (P_1 - P_0')y = \int Q\,dx + c$$

$$\Rightarrow x\frac{dy}{dx} + \left(\frac{2}{x} - 1\right)y = \int \frac{1}{x^2}dx + c$$

$$\Rightarrow \frac{dy}{dx} + \overbrace{\left(\frac{2}{x^2} - \frac{1}{x}\right)}^{P} y = \overbrace{\frac{1}{x}\left[-\frac{1}{x} + c\right]}^{Q} \quad \text{—(iii)}$$

It is linear DE of first order

$$I.f. = e^{\int P dx} = e^{\int \left(\frac{2}{x^2} - \frac{1}{x}\right) dx}$$

$$= e^{-\frac{2}{x} - \log x} = e^{-\frac{2}{x}} e^{\log x^{-1}} = \frac{e^{-2/x}}{x}$$

Sol.^{n} of Eq.^{n} (iii) is

$$y \cdot \left(\frac{e^{-2/x}}{x}\right) = \int \left[\frac{1}{x}\left(-\frac{1}{x} + c\right)\right] \frac{e^{-2/x}}{x} dx + C_1$$

$$= -\int \frac{1}{x^2} \cdot \frac{1}{x} e^{-2/x} dx + c \int \frac{1}{x^2} e^{-2/x} dx + C_1$$

$$\left[\text{let } -\frac{1}{x} = t \Rightarrow +\frac{1}{x^2} dx = dt\right]$$

$$\frac{y}{x} e^{-2/x} = \int t e^{+2t} dt + c \int e^{+2t} dt + C_1$$

$$= \frac{t e^{+2t}}{+2} - \frac{1}{4} e^{2t} + \frac{c}{2} e^{2t} + C_1$$

$$\frac{y}{x} e^{-2/x} = -\frac{1}{2x} e^{-2/x} - \frac{1}{4} e^{-2/x} + \frac{1}{2} c e^{-2/x} + C_2$$

$$\Rightarrow y = -\frac{1}{2} - \frac{x}{4} + \frac{cx}{2} + C_2 x e^{2/x} \quad \text{Ans}$$

Ex:→2. Solve $\sin^2 x \frac{d^2y}{dx^2} = 2y$.

Sol.^{n}→ $\frac{d^2y}{dx^2} - \frac{2}{\sin^2 x} y = 0 \Rightarrow \frac{d^2y}{dx^2} - 2\operatorname{cosec}^2 x \, y = 0$ —(i)

[It is not Exact D.E. & coffecients P_0, P_1 are trigno. fun.^{n}. multiplying it with I.f. = $\cot x$]

8

$$\underbrace{\cot x}_{P_0} \frac{d^2y}{dx^2} + \underbrace{0}_{P_1} \cdot \frac{dy}{dx} + \underbrace{(-2\cot x \,\mathrm{cosec}^2 x)}_{P_2}\, y = \underbrace{0}_{Q} \quad \text{—(ii)}$$

Eq.^{n} (ii) is Exact DE as

$$P_2 - P_1' + P_0'' = -\cancel{2\cot x \,\mathrm{cosec}^2 x} - 0 + \cancel{2\cot x \,\mathrm{cosec}^2 x} = 0$$

Hence first integral of Eq.^{n} (ii)

$$P_0 \frac{dy}{dx} + (P_1 - P_0')\,y = c_1 \quad (\text{as } Q = 0)$$

$$\Rightarrow \cot x \frac{dy}{dx} + (0 + \mathrm{cosec}^2 x)\,y = c_1$$

$$\Rightarrow \frac{dy}{dx} + \frac{\mathrm{cosec}^2 x}{\cot x}\, y = \frac{c_1}{\cot x} \quad \text{—(iii)}$$

It is LDE of first order. &

$$I.f. = e^{\int \frac{\mathrm{cosec}^2 x}{\cot x} dx} = e^{-\log \cot x} = e^{\log(\cot x)^{-1}}$$

$$= \frac{1}{\cot x} = \tan x$$

Hence sol^n of Eq.^{n} (iii) will be

$$y \cdot \tan x = \int \frac{c_1}{\cot x} \cdot \tan x \, dx + c_2$$

$$= c_1 \int (\sec^2 x - 1)\, dx + c_2$$

$$y \cdot \tan x = c_1(\tan x - x) + c_2$$

Assignment :-

1. $x^4 \frac{d^2y}{dx^2} + x^2(x-1)\frac{dy}{dx} + xy = x^3 - 4$

2. $\frac{d^2y}{dx^2} + 2\tan x \frac{dy}{dx} + 3y = \tan^2 x \cdot \sec x$