Ex:- Every field is a vector space over its subfield.

Solⁿ:- let F be a field and H be its subfield. ($F \rightarrow$ set of vectors)
($H \rightarrow$ set of scalars)

let $v_1, v_2 \in F \Rightarrow v_1 + v_2 \in F$ ($\because$ F is a field so $(F,+)$ is an abelian group)

let $a \in H$, $v_1 \in F \Rightarrow a v_1 \in F$ ($\because$ $H \subset F \Rightarrow a \in F$ (subfield) $\therefore a \in F, v_1 \in F \Rightarrow a v_1 \in F$)

(F being a field so it is closed w. r. to multiplication)

(i) $(F,+)$ is an abelian group.

F is a field so by defⁿ.

$\Rightarrow (F,+)$ is an abelian group.

(ii) $\forall\, a \in H$, and $v_1, v_2 \in F$, we have.

$a(v_1 + v_2) = a v_1 + a v_2$ (distributive property holds in F) $a \in H \subset F$

(iii) $\forall\, a, b \in H$ & $v_1 \in F$, we have

$\Rightarrow a, b \in F$ (as $H \subset F$)

$(a+b) \cdot v_1 = a v_1 + b v_1$ (distributive property holds in F)

(iv) $\forall\, a, b \in H$ & $v_1 \in F$

$\Rightarrow a, b \in F$

so $(ab)\alpha = a(b\alpha)$ ($(F, \cdot)$ is an abelian group so holds associativity)

(V) Let 1 be unit element of H

$\Rightarrow$ $1 \cdot v_1 = v_1$, $\forall v_1 \in F$ (as 1 is unit element in F)

$\Rightarrow$ F(H) is a vector space.

H. P.

✱ Since every field 'F' is a subfield of itself. $\Rightarrow$ F(F) is always a vector space.

$\Rightarrow$ C(C), R(R), Q(Q) is a vector space

& C(R), R(Q) are also vector space. (from above)

But R(C), Q(R) is not a vector space.

<u>Theorem 1</u> Let V be a vector space over a field F. If $\mathbf{0}$ (Bold) be the zero vector in V and '0' be the additive identity in F, then:

(i) $a\mathbf{0} = \mathbf{0}$ $\forall a \in F$

(ii) $0\alpha = \mathbf{0}$ $\forall \alpha \in V$

(iii) $a(-\alpha) = -(a\alpha)$, $\forall a \in F, \alpha \in V$

(iv) $(-a)\alpha = -(a\alpha)$, $\forall a \in F, \alpha \in V$

(v) $a(\alpha - \beta) = a\alpha - a\beta$, $\forall a \in F$; $\alpha, \beta \in V$

(vi) for any $a \in F$, $\alpha \in V$

$a\alpha = 0 \Rightarrow a = 0$ or $\alpha = \mathbf{0}$

Proof : (i) $\because$ **0** is the zero vector in V

$\Rightarrow \mathbf{0} + \mathbf{0} = \mathbf{0}$

$\Rightarrow a(\mathbf{0}+\mathbf{0}) = a\mathbf{0} \quad \forall\, a \in F$

$\Rightarrow a\mathbf{0} + a\mathbf{0} = a\mathbf{0}$ ($V(F)$, (ii) property)

$\Rightarrow a\mathbf{0} + a\mathbf{0} = a\mathbf{0} + \mathbf{0}$ (**0** is zero vector in V)

$\Rightarrow a\mathbf{0} = \mathbf{0}$ (left cancellation law)

(ii) since $0 \in F$ is additive identity in F. so

$0 + 0 = 0$

$\Rightarrow (0+0)\alpha = 0\alpha \quad \forall\, \alpha \in V$

$\Rightarrow 0\alpha + 0\alpha = 0\alpha$ ($V(F)$, (iii) property)

$\Rightarrow 0.\alpha + 0.\alpha = 0\alpha + \mathbf{0}$ (**0** is additive identity in V)

(left cancellation law)

$\Rightarrow 0.\alpha = \mathbf{0}$

(iii) for any $a \in F$ & $\alpha \in V$, we have

~~$[a+(-a)]\alpha = a\alpha + (-a)\alpha$ ($V(F)$, (iii) property)~~

$a[\alpha + (-\alpha)] = a\alpha + a(-\alpha)$ ($V(F)$, (ii) property)

$\Rightarrow a\mathbf{0} = a\alpha + a(-\alpha)$

$\Rightarrow \mathbf{0} = a\alpha + a(-\alpha)$ (by (i) proof)

$\Rightarrow a(-\alpha)$ is the additive inverse of $a\alpha$.

$\Rightarrow a(-\alpha) = -(a\alpha)$

(iv) for any. $a \in F$ & $\alpha \in V$,

$[a + (-a)]\alpha = a\alpha + (-a)\alpha$ ($V(F)$, (iii) property)

$\Rightarrow 0.\alpha = a\alpha + (-a)\alpha$

$\Rightarrow \mathbf{0} = a\alpha + (-a)\alpha$ (by (ii) proof)

$\Rightarrow (-a)\alpha$ is additive inverse of $a\alpha$.

$\Rightarrow (-a)\alpha = -(a\alpha)$

(v) for any $a \in F$ & $\alpha, \beta \in V$ we have

$$a(\alpha - \beta) = a[\alpha + (-\beta)]$$
$$= a\alpha + a(-\beta)$$
$$= a\alpha + [-a\beta] \text{ (by (iii) proof)}$$
$$= a\alpha - a\beta$$

(vi) let $a \in F$, $\alpha \in V$ s.t. $a \neq 0$ and $a\alpha = \mathbf{0}$

since every non zero element of F has its ~~multadditive~~ multiplicative inverse in F

$a(\neq 0) \in F \Rightarrow a^{-1} \in F$ ($\because$ F is a field)

since $a\alpha = \mathbf{0} \Rightarrow a^{-1}(a\alpha) = a^{-1}\mathbf{0}$

$\Rightarrow (a^{-1}a)\alpha = \mathbf{0}$ [V(F) (iv) property & (i) proof

$\Rightarrow (1)\alpha = \mathbf{0}$

$\Rightarrow \alpha = \mathbf{0}$ [V(F), (V) property (*)

Now, next let $a\alpha = 0$ and $\alpha \neq 0$,

(then we show that $a = 0$)

If possible, let $a \neq 0 \Rightarrow a^{-1} \in F$

$\because a\alpha = \mathbf{0} \Rightarrow a^{-1}(a\alpha) = a^{-1}\mathbf{0}$

$\Rightarrow (a^{-1}a)\alpha = \mathbf{0}$

[V(F), (iv) property & (i) proof]

$\Rightarrow 1\alpha = 0$

$\Rightarrow \alpha = 0$ [V(F), (V) property]

Which is contradiction of our initial assum- -ption ($\alpha \neq 0$)

Hence $a = 0$ —— (**)

by (*) & (**)

we have $a\alpha = 0 \Rightarrow$ Either $a=0$ or $\alpha = 0$

Theorem :→2 let V(F) be a vector space, then

(i) For any. $a, b \in F$ and $\alpha \in V$, $\alpha \neq 0$

$a\alpha = b\alpha \Rightarrow a = b$

(ii) For any. $\alpha, \beta \in V$ and $a \in F$, $a \neq 0$

$a\alpha = a\beta \Rightarrow \alpha = \beta$

Proof:- (i) $a\alpha = b\alpha \Rightarrow a\alpha - b\alpha = 0$

$\Rightarrow a\alpha + (-b\alpha) = 0$

$\Rightarrow a\alpha + (-b)\alpha = 0$ (Theorem 1)

$\Rightarrow [a + (-b)]\alpha = 0$ [V(F), (iii) prop.]

$\Rightarrow a + (-b) = 0$ [Since $\alpha \neq 0$, Theorem 1 (vi)]

$\Rightarrow a = -(-b)$ (additive inverse)

$\Rightarrow a = b$

(ii) $a\alpha = a\beta \Rightarrow a\alpha - a\beta = 0$

$\Rightarrow a(\alpha - \beta) = 0$ [V(F), (ii) prop.]

$\Rightarrow \alpha - \beta = 0$ [$\because a \neq 0$, Theorem 1 (vi)]

$\Rightarrow \alpha = \beta$

Ex:→ Examine whether the following are vector space or not.

(i) $C(C)$ — $C \to$ complex number

(ii) $R(R)$ — $R \to$ real number

(iii) $C(R)$

(iv) $R(C)$

Solⁿ:→ As we know that Every field is a vector space over its subfield.

$\Rightarrow$ $C(C)$, $R(R)$ is a vector space as C & R are subfield of itself.

Since R is subfield of C.

$\Rightarrow$ $C(R)$ is a vector space.

but $R(C)$ is not a vector space as C is not subfield of R.