$\Rightarrow \alpha = \beta$

Ex:→ Examine whether the following are vector space or not.

(i) $C(C)$ C→ complex number

(ii) $R(R)$ R→ real number

(iii) $C(R)$

(iv) $R(C)$

Sol^n:→ As we know that Every field is a vector space over its subfield.

$\Rightarrow C(C), R(R)$ is a vector space as C & R are subfield of itself.

Since R is subfield of C.

$\Rightarrow C(R)$ is a vector space.

but $R(C)$ is not a vector space as C is not subfield of R.

Ex:→ let V be the set of all ordered pairs of real numbers and F be the field of real numbers. Examine whether $V(F)$ is a vector space or not for the following defined operations:-

(a) $(\alpha_1, \beta_1) + (\alpha_2, \beta_2) = (0, \beta_1 + \beta_2)$

$a(\alpha_1, \beta_1) = (a\alpha_1, a\beta_1)$

(b) $(\alpha_1, \beta_1) + (\alpha_2, \beta_2) = (\alpha_1 + \alpha_2, \beta_1 + \beta_2)$

$a(\alpha_1, \beta_1) = (a^3\alpha_1, a^3\beta_1)$

(c) $(\alpha_1, \beta_1) + (\alpha_2, \beta_2) = (\alpha_2, \beta_2)$

$a(\alpha_1, \beta_1) = (a\alpha_1, a\beta_1)$

(d) $(\alpha_1, \beta_1) + (\alpha_2, \beta_2) = (\alpha_1 + \alpha_2, \beta_1 + \beta_2)$

$a(\alpha_1, \beta_1) = (a\alpha_1, 0)$

$\forall\ a, \alpha_1, \alpha_2, \beta_1, \beta_2 \in R.$

Solⁿ:→ (a) For the ~~diff~~ defined vector addition

$(\alpha_1, \beta_1) + (\alpha_2, \beta_2) = (0, \beta_1 + \beta_2)$

V doesnot contain identity element as

$(\alpha_1, \beta_1) + (\alpha_2, \beta_2) = (0, \beta_1 + \beta_2) \neq (\alpha_1, \beta_1)$

$\Rightarrow$ $(V, +)$ is not abelian group $\Rightarrow$ V(F) is not vector space

(b) For the defined addition operation,

$(V, +)$ is an abelian group but

$(a+b)(\alpha_1, \beta_1) = \{(a+b)^3\alpha_1, (a+b)^3\beta_1\}$

$\neq (a^3\alpha, a^3\beta_1) + (b^3\alpha, b^3\beta_1)$

$\Rightarrow$ (iii) property of vector space (scalar addition is distributive over vector multi.) is not satisfied

$\Rightarrow$ V(F) is not a vector space.

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(iii) For the defined '+' operation, identity and commutative property is not satisfied. as $(\alpha_1, \beta_1) + (\alpha_2, \beta_2) = (\alpha_2, \beta_2)$
$(\alpha_2, \beta_2) + (\alpha_1, \beta_1) = (\alpha_1, \beta_1)$
$\Rightarrow$ Hence $V(F)$ is not a vector space.

(iv) For the defined scalar multiplication, {(v) property of vector space}

$$1 \cdot (\alpha_1, \beta_1) = (1 \cdot \alpha_1, 0) \neq (\alpha_1, \beta_1)$$

Hence $V(F)$ is not a vector space.

Vector subspace (सदिश उपसमष्टि)

Let $V(F)$ be a vector space and W be a non empty subset of V s.t. it is closed for both vector addition and scalar multiplication defined on V. $W(F)$ is said to be subspace of vector space $V(F)$ if it is itself a vector space for the defined vector addition and scalar multiplication.

Every vector space V(F) has at least two subspace (i) V itself.
(ii) {0} zero space

These are known as trivial or improper subspace. Any other subspace except these is known as proper subspace.

Eg:- (i) The set of real numbers $R(R)$ is subspace of $C(R)$

(ii) Set $W = \{(a,b,0) \mid a,b \in R\}$ is subspace of $R^3(R) = \{(a,b,c) \mid a,b,c \in R\}$

Theorem:- The necassary and sufficient conditions for a non empty subset W of a vector space $V(F)$ to be a subspace of $V(F)$ is that W is closed under vector addition and scalar multiplication:

$\alpha_1 \in W,\ \alpha_2 \in W \Rightarrow \alpha_1 + \alpha_2 \in W$ — (i)

$a \in F,\ \alpha \in W \Rightarrow a\alpha \in W$ — (ii)

Proof:→ (necassary condition) ($\Rightarrow$) Let W be a subspace of vector space $V(F)$. $\Rightarrow$ W is itself a vector space over vector addition and scalar multiplication

$\Rightarrow$ W is closed for vector addition and scalar multiplication.

$\Rightarrow$ condition (i) & (ii) are ~~ne~~ satisfied.

(sufficient condition) ($\Leftarrow$) Let W be a non empty subset of V that satisfied condition (i) & (ii)

We prove that W is subspace of $V(F)$

Since $W \neq \phi \Rightarrow \alpha \in W$ & $1 \in F$ is the unit element

$1 \in F \Rightarrow -1 \in F$ (by def.^{n} of F)

$(-1) \in F,\ \alpha \in W \Rightarrow (-1).\alpha \in W$ (by (ii))

$\Rightarrow -(1.\alpha) \in W$ (as $(-a).\alpha = -(a.\alpha)$)

$\Rightarrow -\alpha \in W$ ($\because 1.\alpha = \alpha$)

$\Rightarrow$ Every element of W has its additive inverse in W.

Since $\alpha \in W, -\alpha \in W \Rightarrow \alpha + (-\alpha) \in W$ (by (i))

$\Rightarrow 0 \in W$ (0, zero vector of V)

$\Rightarrow$ zero vector of V is also zero vector of W.

$\Rightarrow W$ contains additive identity.

Since $W \subset V \Rightarrow W$ holds associativity and commutativity for addition as it is hold in V.

Hence $(W, +)$ is an abelian group.

by condition (ii), W is closed w. r. to scalar multiplication. & $W \subset V$

$\Rightarrow W$ holds all the remainning properties [(ii),(iii),(iv),(v)] of vector space.

$\Rightarrow W$ is subspace of vector ~~sc~~ space $V(F)$