Theorem:-2 The necassary and sufficient conditions for a non void subset W of a vector space $V(F)$ to be a subspace of $V(F)$ are

(i) $\alpha \in W, \beta \in W \Rightarrow (\alpha - \beta) \in W$

(ii) $a \in F, \alpha \in W \Rightarrow a\alpha \in W$

Proof:→ (necassary condition) ($\Rightarrow$) let W be a subspace of vector space $V(F)$.

$\Rightarrow$ W is itself a vector space for vector addition & scalar multiplication defined on V

$\Rightarrow$ $(W, +)$ is an abelian group.

$\alpha \in W, \beta \in W \Rightarrow \alpha \in W, -\beta \in W$ (additive inverse)

$\Rightarrow \alpha + (-\beta) \in W$ (closure prop.)

$\Rightarrow (\alpha - \beta) \in W$

$\Rightarrow$ condition (i) is necassary.

since W is closed w.r. to scalar multipli.

$a \in F, \alpha \in W \Rightarrow a\alpha \in W$

Hence condition (ii) is also necassary.

(Sufficient Condition) ($\Leftarrow$):-let W be a non empty subset of V s.t. it satisfy conditions (i) & (ii). We prove that W is subspace of V.

Since $W \neq \phi \Rightarrow \alpha \in W$ so by (i)

$\alpha \in W, \alpha \in W \Rightarrow \alpha - \alpha \in W$

$\Rightarrow 0 \in W$

$\Rightarrow$ W contains zero vector (additive identity)

Since $0 \in W, \alpha \in W \Rightarrow 0 - \alpha \in W$ (by (i))

$\Rightarrow -\alpha \in W$

$\Rightarrow$ every element of W has its additive inverse in W.

$\alpha \in W, \beta \in W \Rightarrow \alpha \in W, -\beta \in W$ (by above)

$\Rightarrow \alpha - (-\beta) \in W$ (by (i))

$\Rightarrow \alpha + \beta \in W$

$\Rightarrow$ W is closed for vector addition & by (ii) it is closed for scalar multiplication.

So by previous theorem (Write the statement)

$\Rightarrow$ W is subspace of vector space V(F).

Theorem :- 3 The necassary and sufficient condition for a non void subset W of a vector space V(F) to be subspace of V(F) is

$a, b \in F$ and $\alpha, \beta \in W \Rightarrow a\alpha + b\beta \in W$ —(i)

Proof :→ (<u>necassary</u> <u>condition</u>) ($\Rightarrow$)

(21) let W be a subspace of vector space $V(F)$, then we show that condition (i) is satisfied.

since W is subspace $\Rightarrow$ W is closed w.r.to scalar multiplication.

so $a \in F, \alpha \in W \Rightarrow a\alpha \in W$

$b \in F, \beta \in W \Rightarrow b\beta \in W$

Now $a\alpha \in F, b\beta \in W \Rightarrow a\alpha + b\beta \in W$ (W is closed w.r. to vector addition)

Hence condition (i) is necassary.

Sufficient condition ($\Leftarrow$):→ let W be a non empty subset of vector space $V(F)$ s.t. it satisfy. condition (i). We prove that W is subspace of $V(F)$.

Taking $a = b = 1$ in condition (i)

$1, 1 \in F$ (unity Element) & $\alpha, \beta \in W \Rightarrow (1\alpha + 1\beta) \in W$

$\Rightarrow \alpha + \beta \in W$

$\Rightarrow$ W is closed w.r. to vector addition

$1 \in F \Rightarrow -1 \in F$ (by defn of F)

Again $-1 \in F, 0 \in F$ and $\alpha, \beta \in W$ (by condition (i))

$\Rightarrow (-1)\alpha + 0 \cdot \beta \in W$

$\Rightarrow -\alpha \in W$

$\Rightarrow$ every element of W has its additive inverse in W.

Now $a=0, b=0 \in F$ & $\alpha, \beta \in W \Rightarrow 0.\alpha + 0.\beta \in W$

$\Rightarrow 0+0 \in W$

$\Rightarrow 0 \in W$

$\Rightarrow$ W contains zero vector (additive identity)

since $W \subset V \Rightarrow$ W holds associative and commutative property for vector addition as it holds in V.

Hence $(W,+)$ is an abelian group.

Taking $a, b(=0) \in F$ & $\alpha, \beta \in W$

$\Rightarrow a.\alpha + 0.\beta \in W$ (by condition (i))

$\Rightarrow a\alpha + 0 \in W$

$\Rightarrow a\alpha \in W$

$\Rightarrow$ W is closed w.r. to scalar multiplication.

Since $W \subset V \Rightarrow$ W Holds all the remainning properties {(ii), (iii), (iv), (v)} of vector space as it holds closure property for vector addition and scalar multiplication.

$\Rightarrow$ W is subspace of vector space V(F).

Ex→1. Examine which of the following sets is a subspace of the vector space $V_3(R)$

(i) $W = \{(\alpha, \beta, \gamma) \mid \alpha + 2\beta + 3\gamma = 0 ; \alpha, \beta, \gamma \in R\}$

(ii) $W = \{(\alpha, \beta, \gamma) \mid \alpha, \beta, \gamma \in Z\}$

Sol^n:→ (i) let $v_1, v_2 \in W$

$\Rightarrow v_1 = (\alpha_1, \beta_1, \gamma_1)$ & $v_2 = (\alpha_2, \beta_2, \gamma_2) \in W$

where $\alpha_1 + 2\beta_1 + 3\gamma_1 = 0$, $\alpha_2 + 2\beta_2 + 3\gamma_2 = 0$ — (i)

let $a, b \in R$, & $v_1, v_2 \in W$

$$av_1 + bv_2 = a(\alpha_1, \beta_1, \gamma_1) + b(\alpha_2, \beta_2, \gamma_2)$$

$$= (a\alpha_1, a\beta_1, a\gamma_1) + (b\alpha_2, b\beta_2, b\gamma_2)$$

$$= (a\alpha_1 + b\alpha_2, a\beta_1 + b\beta_2, a\gamma_1 + b\gamma_2)$$

$$= (a(-2\beta_1 - 3\gamma_1) + b(-2\beta_2 - 3\gamma_2), a\beta_1 + b\beta_2, a\gamma_1 + b\gamma_2)$$

$$= (\underbrace{-2(a\beta_1 + b\beta_2) - 3(a\gamma_1 + b\gamma_2)}_{\alpha_3}, \underbrace{a\beta_1 + b\beta_2}_{\beta_3}, \underbrace{a\gamma_1 + b\gamma_2}_{\gamma_3})$$

$\in W$

{as $\alpha_3 + 2\beta_3 + 3\gamma_3 = 0$} Hence by theorem 3.,

W is subspace of vector space $V_3(R)$

(ii) let $v_1 \in W$ where $v_1 = (\alpha_1, \beta_1, \gamma_1)$

& $\alpha_1, \beta_1, \gamma_1 \in Z$

let $a = \frac{1}{3} \in R$ & $v_1 \in W$

$$av_1 = \frac{1}{3}(\alpha_1, \beta_1, \gamma_1) = \left(\frac{\alpha_1}{3}, \frac{\beta_1}{3}, \frac{\gamma_1}{3}\right) \notin W$$

as $\frac{\alpha_1}{3}, \frac{\beta_1}{3}, \frac{\gamma_1}{3}$ are all not integers.

Hence W is not closed w.r.to scalar multiplication

$\Rightarrow$ W is not subspace of V(F).

Ex: If $V(F)$ is a vector space of all $n \times n$ matrices over the field F, then prove that the set W of all $n \times n$ symmetric matrices over F will be a subspace of $V(F)$. (24)

Solⁿ:- let $A, B \in W \Rightarrow A^T = A$, $B^T = B$ (i) (given)

let $a, b \in F$ & $A, B \in W \subset V \Rightarrow a, b \in F$ & $A, B \in V$

$\Rightarrow aA \in V$, $bB \in V$

$\Rightarrow aA + bB \in V$

(by defin. of vector space)

$$(aA + bB)^T = (aA)^T + (bB)^T \quad \text{(from transpose property)}$$
$$= aA^T + bB^T$$
$$= aA + bB \quad \text{(by (i))}$$

$\Rightarrow aA + bB$ is also a symmetric matrix of V

$\Rightarrow aA + bB \in W$

so $a, b \in F$ & $A, B \in W \Rightarrow aA + bB \in W$

$\Rightarrow W$ is subspace of $V(F)$.