Union of two subspace :-

let $V_3(F)$ be a vector space.

$$V_3(F) = \{(a,b,c) \mid a,b,c \in R\}$$

let $W_1 = \{(a,0,0) \mid a \in R\}$

& $W_2 = \{(0,b,0) \mid b \in R\}$ are two subspace of $V(F)$

$$W_1 \cup W_2 = \{\alpha \mid \alpha = (a,0,0) \text{ or } (0,b,0)\}$$

$\alpha, \beta \in W_1 \cup W_2$ let $\alpha = (a,0,0)$

& $\beta = (0,b,0)$, $p, q \in F$

$$p\alpha + q\beta = p(a,0,0) + q(0,b,0)$$

$$= (pa, 0, 0) + (0, qb, 0)$$

$$= (pa, qb, 0) \notin W_1 \cup W_2$$

$\Rightarrow$ $W_1 \cup W_2$ is not a subspace of $V_3(F)$.

Theorem:- The union of two subspace W_1 & W_2 of a vector space $V(F)$ is a subspace if and only if either $W_1 \subset W_2$ or $W_2 \subset W_1$

Proof:- ($\Rightarrow$) let W_1 and W_2, $W_1 \cup W_2$ are two subspace of vector space. then we prove that $W_1 \subset W_2$ or $W_2 \subset W_1$.

let $W_1 \not\subset W_2 \Rightarrow \exists\ \alpha_1 \in W_1$ s.t. $\alpha_1 \notin W_2$ —(i)

let $\alpha_2 \in W_2$ then $\alpha_1 \in W_1 \cup W_2$ & $\alpha_2 \in W_1 \cup W_2$

$\Rightarrow \alpha_1 + \alpha_2 \in W_1 \cup W_2$ [$\because W_1 \cup W_2$ is subspace]

$\Rightarrow \alpha_1 + \alpha_2 \in W_1$ or $\alpha_1 + \alpha_2 \in W_2$

$\alpha_1 + \alpha_2 \in W_2$, $\alpha_2 \in W_2 \Rightarrow [(\alpha_1 + \alpha_2) - \alpha_2] \in W_2$

$\Rightarrow \alpha_1 \in W_2$ —(ii)

which is contradiction of Eq.^{n} (i), Hence

$\alpha_1 + \alpha_2 \notin W_2 \Rightarrow \alpha_1 + \alpha_2 \in W_1$

$\alpha_1 + \alpha_2 \in W_1$, $\alpha_1 \in W_1 \Rightarrow (\alpha_1 + \alpha_2) - \alpha_1 \in W_1$

$\Rightarrow \alpha_2 \in W_1$

$\Rightarrow W_2 \subset W_1$

Hence. if

$W_1 \cup W_2$ is a subspace of $V(F)$ & $W_1 \not\subset W_2$

$\Rightarrow W_2 \subset W_1$

conversly :- let W_1 & W_2 be two subspace of vector space $V(F)$ s.t. either $W_1 \subset W_2$ or

$W_2 \subset W_1$ then

$$W_1 \cup W_2 = \begin{cases} W_1 & \text{when } W_2 \subset W_1 \\ W_2 & \text{when } W_1 \subset W_2 \end{cases}$$

$\Rightarrow$ $W_1 \cup W_2 = W_1$ or W_2 & since both of these are subspace of $V(F)$

$\Rightarrow$ $W_1 \cup W_2$ is also a subspace of $V(F)$.