Th. Show that, if a function is continous on $[a, b]$ then it is bounded in that interval.

Proof:- It is given that f is continous in the interval $[a, b]$, then we know that the interval $[a, b]$ can be subdivided into finite number of closed subintervals

$[a = a_0, a_1], [a_1, a_2], [a_2, a_3] - - - [a_{n-1}, a_n = b]$

s.t. $|f(x_1) - f(x_2)| < \epsilon$ ---- (i)

whence x_1 & x_2 belongs to the same subinterval

let $x \in [a, b]$ is point, which lies in subinterval $[a_{i-1}, a_i]$:

consider x & a_{i-1} two points, which are different ~~...~~. Then from (i)

$$|f(x) - f(a_{i-1})| < \epsilon \quad \text{——— (2)}$$

Now, $|f(x)| = |f(x) - f(a_{i-1}) + f(a_{i-1})|$

$$\leq |f(x) - f(a_{i-1})| + |f(a_{i-1})|$$

$$\Rightarrow |f(x)| \leq \epsilon + |f(a_{i-1})| \quad \text{——— (3)}$$

In special case, take $x = a_i$

$$\Rightarrow |f(a_i)| < \epsilon + |f(a_{i-1})|$$

$$\Rightarrow \sum |f(a_i)| < \sum \epsilon + \sum |f(a_{i-1})|$$

$$\sum_{l=1}^{i-1} |f(a_l)| < \sum_{l=1}^{i-1} \epsilon + \sum_{l=1}^{i-1} |f(a_{l-1})|$$

$\Rightarrow |f(a_1)| + |f(a_2)| + \cdots |f(a_{i-1})|$
$< \{\epsilon + \epsilon + \epsilon \cdots (i-1) \text{ times}\} + \{|f(a_0)|$
$+ |f(a_1)| + \cdots |f(a_{i-2})|\}$

$\Rightarrow |f(a_{i-1})| < |f(a_0)| + (i-1)\epsilon$

Here $a_0 = a$

$\Rightarrow |f(a_{i-1})| < f(a) + (i-1)\epsilon$ —— (4)

Adding (3) & (4), we have

$|f(x)| < |f(a)| + i\epsilon \quad \forall x \in [a_{i-1}, a_i]$ —— (5)

Similarly for the sub interval $[a_{n-1}, a_n]$
we have $|f(x)| < |f(a)| + n\epsilon$

$\Rightarrow \forall x \in [a, b]$
$|f(x)| < |f(a)| + n\epsilon$

or $f(a) - n\epsilon < f(x) < f(a) + n\epsilon$
$\forall x \in [a, b]$

$\Rightarrow f(x)$ is bounded in the interval $[a, b]$

H.P.

Th. If a function f is continous in [a,b] then show that it attains it's supremum and infimum at least once in [a,b].

Proof:– It is given that f is continous in [a,b]

⇒ It will be bounded in [a,b] also.

⇒ It will have it's minima & maxima.

let they are m & M respectively.

Then we are to prove that, ∃ at least two numbers ~~in~~ α & β in [a,b], s.t.

$$f(\alpha) = m \quad \& \quad f(\beta) = M.$$

Now, ~~for this purpose~~, let us suppose that $f(x) \neq M \quad \forall x \in [a,b]$

[which means that, no point exist in [a,b] ~~for~~ which $f(x) = M$]

$$\Rightarrow \quad M - f(x) \neq 0 \quad \forall x \in [a,b]$$

let $\phi(x) = \dfrac{1}{M - f(x)}$

Since $M - f(x) \neq 0$ & $M - f(x)$ will also be continous in [a,b]

$\Rightarrow$ $\phi(x)$ will also be continous in [a,b]

$\Rightarrow$ $\phi(x)$ will be bounded in [a,b]

let U be the upper bound of f in [a,b]

$\Rightarrow \quad \phi(x) \leq U \quad \forall\, x \in [a, b]$

$\Rightarrow \quad \dfrac{1}{M - f(x)} \leq U \quad \forall\, x \in [a, b]$

$\Rightarrow \quad M - f(x) \geq \dfrac{1}{U}$

$\Rightarrow \quad f(x) \leq M - \dfrac{1}{U}$

$\Rightarrow \quad f(x) < M \quad \forall\, x \in [a, b]$

But M was the supremum for $f(x)$ in $[a, b]$, means no element/number exists less than M, who can be an upper bound of $f(x)$ in $[a, b]$

$\Rightarrow$ $f(x)$ cannot be less than $M - \frac{1}{U}$

$\Rightarrow$ our original assumption was wrong.

$\Rightarrow \quad f(x) = M$ for some $x \in [a, b]$

$\Rightarrow$ ~~Supremum lies~~

$f(x)$ attains it's supremum at least once in $[a, b]$

Similar we can prove it for minima / infimum.

H.P.