Points to be noted: (i) It is not necessary that every sequence has a limit point.
(ii) There may be unique limit point
(iii) There may be finite or infinite limit point.

Ex (i) If $\langle x_n \rangle = \langle \frac{1}{3} \rangle = \{1/3, 1/3, 1/3, \ldots\}$

or $x_n = \frac{1}{3}\ \forall n \in N$, then $\frac{1}{3}$ is a unique limit point of $\langle x_n \rangle$.

Ex (ii) $\langle x_n \rangle = \langle (-1)^n \rangle = \langle -1, 1, -1, 1, -1, 1 \ldots \rangle$

has -1 & 1 limit points.

Ex (iii) $\langle \frac{(-1)^n}{n} \rangle$ has a limit point '0'.

Th **Bolzano Weierstrass theorem**

Statement: Prove that every bounded sequence has at least one limit point.

Proof:- Let $\langle x_n \rangle$ be a bounded sequence and E is the range set of $\langle x_n \rangle$.

$\Rightarrow \quad E = \{x_n : n \in N\}$

There, here two cases arise (i) E may be finite
(ii) E may be infinite

case (i) when E is finite:

$\Rightarrow$ in this situation, $\exists$ a real number α so that for infinite values of $n \in N$,

$x_n = \alpha$

$\Rightarrow$ $x_n = \alpha$ for infinite values of $n \in N$

$\Rightarrow$ $x_n = \alpha \in (\alpha - \delta, \alpha + \delta)$ for infinite $n \in N$

$\Rightarrow$ $(\alpha - \delta, \alpha + \delta)$ contains α for infinite $n \in N$

$\Rightarrow$ α is a limit point of $\langle x_n \rangle$.

Case (ii) when E is an infinite set.

But it is given that $\langle x_n \rangle$ is bounded

$\Rightarrow$ E will also be bounded

$\Rightarrow$ E is an infinite bounded set $\in R$.

But as per theorem on set, we know that every infinite bounded set has at least one limit point.

$\Rightarrow$ E has at least one limit point

$\Rightarrow$ Let α be the limit point of E.

$\Rightarrow$ Every n.b.d. of α contains infinite members of E

$\Rightarrow$ $(\alpha - \delta, \alpha + \delta)$ contains infinite members of E

$\Rightarrow$ $(\alpha - \delta, \alpha + \delta)$ contains infinite members of $\langle x_n \rangle$.

$\Rightarrow$ α is also a limit point of $\langle x_n \rangle$.

H.P.

Limit of a sequence

Let $\langle x_n \rangle$ be a real sequence and l be a real number. Then l is called limit of $\langle x_n \rangle$ if for a given $\epsilon > 0$ $\exists$ a positive integer $n_0 \in \mathbb{N}$ (depends on ϵ) s.t.

$$|x_n - l| < \epsilon \quad \forall\, n > n_0$$

i.e. after some stage n_0, the difference of x_n and l can be made as small as we please. Symbolically. $\lim\limits_{n \to \infty} \langle x_n \rangle = l$.

Note:
Here it must be clear to all that limit of a sequence is also be the limit point of the sequence but it's converse is not necessarily be true.

Ex $\langle (-1)^n \rangle$ has two limit points $-1, 1$ but limit of $\langle (-1)^n \rangle$ doesnot exists.
because $\lim\limits_{n \to \infty} (-1)^n = 1, -1$ if n is even & n is odd

$\Rightarrow \lim\limits_{n \to \infty} \langle (-1)^n \rangle$ doesnot exists.

Convergencent Sequence

A sequence $\langle x_n \rangle$ is said to be convergent if $\lim\limits_{n \to \infty} x_n$ is finite.

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