Th. If $\langle x_n \rangle$ and $\langle y_n \rangle$ are two sequences such that $x_n \leq y_n \ \forall n \in N$ and $\lim x_n = \xi$ and $\lim y_n = \eta$, then show that $\xi \leq \eta$

Proof:- Since $x_n \leq y_n \quad \forall n \in N$

$\Rightarrow y_n \geq x_n \Rightarrow y_n - x_n \geq 0$

$\Rightarrow \lim (y_n - x_n) \geq 0$

$\Rightarrow \lim y_n \geq \lim x_n \quad \forall n \in N.$

$\Rightarrow \quad \eta \geq \xi \quad$ H.P.

Th. If a sequence $\langle x_n \rangle$ is convergent then show that $\langle |x_n| \rangle$ is also convergent but it's converse is not true.

Proof:- let $\langle x_n \rangle$ is convergent & $\lim x_n = \xi$

Now, $|x_n| = \begin{cases} x_n & \text{if } x_n \geq 0 \\ -x_n & \text{if } x_n \leq 0 \end{cases}$

If $x_n \geq 0 \Rightarrow x_n = |x_n|$

$\Rightarrow \lim x_n = \lim |x_n| = \xi \Rightarrow \langle x_n \rangle$ is convergt.

Again $x_n < 0 \Rightarrow |x_n| = -x_n$

$\Rightarrow \lim |x_n| = -\lim x_n = -\xi$

$\Rightarrow$ ~~$\langle x_n \rangle$~~ $\langle |x_n| \rangle$ is also convergent.

$\Rightarrow \langle |x_n| \rangle$ is convergent.

Q1. Prove that the sequence $\langle n^{1/n} \rangle$ converges to 1.

Sol^n. Let $x_n = n^{1/n} = 1 + a_n$ where $a_n > 0$ $\forall n \in N$.

$\Rightarrow n = (1+a_n)^n$

$\Rightarrow n = 1 + n.a_n + \frac{n(n-1)}{\lfloor 2} a_n^2 + \cdots a_n^2.$

$\Rightarrow n > \frac{n(n-1)}{\lfloor 2} a_n^2$

$\Rightarrow \frac{2}{(n-1)} > a_n^2$ Hence $n > 1$ or $n \geq 2$

$\Rightarrow |a_n| < \sqrt{\frac{2}{n-1}}$

Now if $\sqrt{\frac{2}{n-1}} < \epsilon$ (let)

$\Rightarrow |a_n| < \epsilon$ with $\sqrt{\frac{2}{n-1}} < \epsilon$

$\Rightarrow n-1 > \frac{2}{\epsilon^2}$

$\Rightarrow n > 1 + \frac{2}{\epsilon^2}$

It means that whenever $n > 1 + \frac{2}{\epsilon^2}$

$|a_n| < \epsilon$

$\Rightarrow$ when $\epsilon > 0$, $\exists\, n_0 \left(> + \frac{2}{\epsilon^2} \right) \in N$

s.t. $|a_n| < \epsilon$ $\forall n > n_0$

$\Rightarrow |x_n - 1| < \epsilon$ $\forall n > n_0$

$\Rightarrow \lim x_n = 1$

$\Rightarrow \lim_{n \to \infty} (n)^{1/n} = 1 \Rightarrow \langle n^{1/n} \rangle$ converges to 1.

Q. Show that $\langle \frac{1}{n} \rangle$ converges to 0 by definition.

Sol^n. Let $\epsilon > 0 \Rightarrow \exists\, n_0 \in N$ s.t.

$n_0 \epsilon > 1$

$\Rightarrow \frac{1}{n_0} < \epsilon$

Now for $n > n_0$

i.e. $\frac{1}{n} < \frac{1}{n_0} < \epsilon$

$\Rightarrow \quad x_n < \frac{1}{n_0} < \epsilon$

$\Rightarrow \quad x_n < \epsilon \quad \forall\, n > n_0$

$\Rightarrow \quad |x_n - 0| < \epsilon \quad \forall n > n_0$

$\Rightarrow \quad \lim_{n \to \infty} x_n = 0$

$\Rightarrow \quad \langle x_n \rangle$ converges to 0. H.P.

Another Method:—

Take $x_{n+1} - x_n = \frac{1}{n+1} - \frac{1}{n}$

$= \frac{-1}{n(n+1)} < 0 \quad \forall n \in N$

$\Rightarrow \langle x_n \rangle$ is a decreasing sequence.

and $x_n = \frac{1}{n} > 0$ and $\left\{\frac{1}{n}\right\}$ set is bounded from below.

$\Rightarrow \langle x_n \rangle$ sequence is decreasing & bounded from below

$\Rightarrow \langle x_n \rangle$ is convergent & $\lim x_n = \inf \langle x_n \rangle = 0$

H.P.

Q. Prove that the sequence $\langle x_n \rangle$, where $x_n = \{\sqrt{n+1} - \sqrt{n}\}$ $\forall n \in N$ is convergent.

Sol.

Take $x_n = \left(\sqrt{n+1} - \sqrt{n}\right)\left(\frac{\sqrt{n+1} + \sqrt{n}}{\sqrt{n+1} + \sqrt{n}}\right)$

$$= \frac{1}{\sqrt{n+1} + \sqrt{n}}$$

$$< \frac{1}{\sqrt{n} + \sqrt{n}} = \frac{1}{2\sqrt{n}}$$

$\Rightarrow x_n < \frac{1}{2\sqrt{n}}$

$\Rightarrow |x_n - 0| < \frac{1}{2\sqrt{n}}$, if take $\frac{1}{2\sqrt{n}} < \epsilon$

then $|x_n - 0| < \epsilon$ with $\frac{1}{2\sqrt{n}} < \epsilon$ for $\epsilon > 0$

$\Rightarrow$ for $\epsilon > 0$, $\exists$ n_0 (which is $> \frac{1}{4\epsilon^2}$) $\in N$ s.t.

$\Rightarrow |x_n - 0| < \epsilon \quad \forall n > n_0$

$\Rightarrow \lim_{n \to \infty} x_n = 0$

$\Rightarrow \langle x_n \rangle$ is a convergent sequence.