UNIT - II

t-distⁿ :- In probability and Statistics, Student's t-distribution or t-distⁿ is any member of a family of continuous prob. distⁿ that arises when estimating the mean of a normally distributed popⁿ in situations where the sample size is small and the popⁿ standard deviation is unknown. It was developed by William Sealy Gosset (W.S. Gosset) under the pen name Student.

Defⁿ Student's 't' :- Let x_i, $(i = 1, 2, \ldots, n)$ be a random sample of size n from a normal population with mean μ and variance σ^2. Then Student's t is defined by the statistic

$$t = \frac{\bar{x} - \mu}{s/\sqrt{n}}$$

where

$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$, is the sample mean and

$s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2$ is unbiased estimate of the popⁿ variance σ^2, and it follows Student's t-distⁿ with $\nu = (n-1)$ d.f. with prob. density function (p.d.f.).

$$f(t) = \frac{1}{\sqrt{\nu}\, B\left(\frac{1}{2}, \frac{\nu}{2}\right)} \frac{1}{\left[1 + \frac{t^2}{\nu}\right]^{\frac{\nu+1}{2}}} \; ; \quad -\infty < t < \infty \qquad (*)$$

Remark (1) A statistic t following Student's t-distⁿ with n d.f. will be abbreviated as $t \sim t_n$

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(2) If we take $\nu = 1$, we get

$$f(t) = \frac{1}{B(\frac{1}{2}, \frac{1}{2})} \frac{1}{(1+t^2)}$$

$$= \frac{1}{\pi} \cdot \frac{1}{(1+t^2)} \quad ; \quad -\infty < t < \infty$$

which is p.d.f. of standard Cauchy distribution
Hence when $\nu = 1$ student's t- distn. reduces to Cauchy distn.

Derivation of Student's t-distn. :-

The expression
$$t = \frac{\bar{x} - \mu}{S/\sqrt{n}}$$

can be re-written as

$$t^2 = \frac{n(\bar{x}-\mu)^2}{S^2} = \frac{n(\bar{x}-\mu)^2}{ns^2/n-1}$$

as sample variance $s^2 = \frac{1}{n}\Sigma(x_i - \bar{x})^2$ & ~~popn~~ unbiased estimate of popn variance $S^2 = \frac{1}{n-1}\Sigma(x_i - \bar{x})^2$

$\Rightarrow ns^2 = (n-1)S^2$

Since x_i, $(i = 1, 2, -, n)$ is a r.s. from the normal popn with mean μ and variance σ^2.

$$\bar{x} \sim N(\mu, \sigma^2/n)$$

$$\Rightarrow \frac{\bar{x} - \mu}{\sigma/\sqrt{n}} \sim N(0,1)$$

Hence $\left(\frac{(\bar{x}-\mu)^2}{\sigma^2/n}\right)$ being the square of standard normal variate is a chi-square variate with 1 d.f.

Also $\frac{ns^2}{\sigma^2}$ is a χ^2 variate with $(n-1)$ d.f.

Further since $\bar{x}$ and s^2 are indeptly distributed.

$\frac{t^2}{n-1}$ being the ratio of two indept. χ^2-variate with 1 d.f and $(n-1)$ d.f. respectively, is a $\beta_2\left(\frac{1}{2}, \frac{n-1}{2}\right)$ variate and its distribution

is given by

$$dF(t) = \frac{1}{B\left(\frac{1}{2}, \frac{\nu}{2}\right)} \frac{\left(\frac{t^2}{\nu}\right)^{\frac{1}{2}-1}}{\left(1+\frac{t^2}{\nu}\right)^{\frac{\nu+1}{2}}} d(t^2/\nu) \qquad 0 \le t^2 < \infty$$

where $\nu = n-1$

$$= \frac{1}{\sqrt{\nu}\, B\left(\frac{1}{2}, \frac{\nu}{2}\right)} \frac{1}{\left(1+\frac{t^2}{\nu}\right)^{\frac{\nu+1}{2}}} dt \qquad -\infty < t < \infty$$

the factor 2 disappearing since the integral from $-\infty$ to ∞ must be unity. Thus the above prob. function is required p.d.f. of Student's t-distn. with $\nu = (n-1)$ d.f.

Remark :-

Confidence or Fiducial limits for μ :- If $t_{0.05}$ is the tabulated value of t for $\nu = (n-1)$ d.f. at 5% l.o.s.

i.e.

$$P(|t| > t_{0.05}) = 0.05$$

$$\Rightarrow P(|t| \le t_{0.05}) = 0.95$$

the 95% confidence limits for μ are given by

$$|t| \le t_{0.05} \quad \text{i.e.} \quad \left|\frac{\bar{x}-\mu}{s/\sqrt{n}}\right| \le t_{0.05}$$

$$\left(\bar{x} - t_{0.05} \cdot \frac{S}{\sqrt{n}}\right) \le \mu \le \left(\bar{x} + t_{0.05} \frac{S}{\sqrt{n}}\right)$$

Thus, 95% confidence limits for μ are :

$$\bar{x} \pm t_{0.05} \cdot \frac{S}{\sqrt{n}}$$

Similarly 99% confidence limits for μ are :

$$\bar{x} \pm t_{0.01} \frac{S}{\sqrt{n}}$$

where $t_{0.01}$ is the tabulated value of t for $\nu = (n-1)$ d.f. at 1% l.o.s.

Fisher's t :- Student's t-dist^{n} is later developed and extended by Prof. R.A. Fisher (1926).

Def^{n} :- It is the ratio of a standard normal variate to the square root of an indept. chi-square variate divided by its degrees of freedom.

If ξ is a $N(0,1)$ and χ^2 is an indept. chi-square variate with n d.f., then Fisher's t is given by

$$t = \xi \Big/ \sqrt{\frac{\chi^2}{n}}$$

and it follows student's 't' distribution with n degrees of freedom.

Distn. of Fisher's 't' :—

Since ξ and χ^2 are indept, their joint prob. differential is given by

$$dF(\xi, \chi^2) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{\xi^2}{2}\right) \frac{\exp(-\chi^2/2)(\chi^2)^{\frac{n}{2}-1}}{2^{n/2}\,\Gamma n/2}\, d\xi\, d\chi^2$$

Let us transform to new variates t and u by the substitution

$$t = \frac{\xi}{\sqrt{\chi^2/n}} \quad \text{and} \quad u = \chi^2$$

$$\Rightarrow \xi = t\sqrt{u/\sqrt{n}} \quad \& \quad \chi^2 = u$$

Jacobian of transformation J is given by

$$J = \frac{\partial(\xi, \chi^2)}{\partial(t, u)} = \begin{vmatrix} \sqrt{u/n} & t/2\sqrt{un} \\ 0 & 1 \end{vmatrix} = \sqrt{\frac{u}{n}}$$

The joint distn. of t and u becomes

$$dG(t,u) = \frac{1}{\sqrt{2\pi}\, 2^{n/2}\, \Gamma n/2} \exp\left\{-\frac{u}{2}\left(1+\frac{t^2}{n}\right)\right\} \times (u)^{\frac{n}{2}-1} \times \sqrt{\frac{u}{n}}\, du\, dt$$

integrating w.r.t. 'u' over the range 0 to ∞, the marginal distn. of t becomes

$$dG_1(t) = \frac{1}{\sqrt{2\pi}\, 2^{n/2}\, \Gamma n/2\, \sqrt{n}} \left[\int_0^\infty \exp\left\{-\frac{u}{2}\left(1+\frac{t^2}{n}\right)\right\} u^{\frac{n}{2}-\frac{1}{2}} du\right] dt$$

$$= \frac{1}{\sqrt{2\pi}\, 2^{n/2}\, \Gamma n/2\, \sqrt{n}} \frac{\Gamma(n+1)/2}{\left[\frac{1}{2}\left(1+\frac{t^2}{n}\right)\right]^{\frac{n+1}{2}}} dt$$

As $\int_0^\infty e^{-ax} x^{n-1} dx = \frac{\Gamma n}{a^n}$ is a gamma variate

$$\therefore dv_1(t) = \frac{\Gamma(n+1)/2}{\sqrt{n}\,\Gamma\frac{n}{2}\,\Gamma\frac{1}{2}} \cdot \frac{1}{\left(1+\frac{t^2}{n}\right)^{\frac{n+1}{2}}} dt \quad ; -\infty < t < \infty$$

$$= \frac{1}{\sqrt{n}} \cdot \frac{1}{B\left(\frac{1}{2}, \frac{n}{2}\right)} \frac{1}{\left(1+\frac{t^2}{n}\right)^{\frac{n+1}{2}}} dt$$

$$-\infty < t < \infty$$

which is same as the prob. funⁿ of Student's t-distⁿ. with n d.f.

Remark : (1) In Fisher's 't' the d.f. is the same as the d.f. of chi-square variate.

(2) Student's 't' may be regarded as a particular case of Fisher's 't' as explained below.

Since $\bar{x} \sim N(\mu, \sigma^2/n)$

$$\xi = \frac{\bar{x}-\mu}{\sigma/\sqrt{n}} \sim N(0,1)$$

and $$\chi^2 = \frac{ns^2}{\sigma^2} = \sum_{i=1}^{n} (x_i - \bar{x})^2/\sigma^2$$

is indeptly distributed as chi-square variate with $(n-1)$ d.f. Hence Fisher's t is given by

$$t = \frac{\xi}{\sqrt{\chi^2/(n-1)}} = \frac{\sqrt{n}(\bar{x}-\mu)}{\sigma} \cdot \frac{\sigma}{\sqrt{\Sigma(x_i-\bar{x})^2/(n-1)}}$$

$$= \frac{\sqrt{n}(\bar{x}-\mu)}{S} = \frac{\bar{x}-\mu}{S/\sqrt{n}}$$

and it follows student's t-distⁿ. with $(n-1)$ d.f.