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χ^2 - Distribution :- (Pronounced as Ki, sky without s is a Greek alphabet)

For small sample, the sampling distribution of the statistic in its standard form

viz. $$Z = \frac{t - E(t)}{S.E(t)}$$; where t is a statistic of sample

is not normal, in such cases we apply exact sampling test which depends upon the magnitude of n. Here we discussed about the chi-square distribution.

The square of a standard normal variable is called a chi-square variate with 1 d.f.. Thus if X is a random variable following normal distribution with mean μ and variance σ^2 ($s.d. = \sigma$) then

$\frac{X-\mu}{\sigma}$ is a standard normal variate.

$\therefore$ $\left(\frac{X-\mu}{\sigma}\right)^2$ is a chi-square variate with 1 d.f.

In general if X_i ($i = 1, 2, \dots n$) are n indept. normal variates with mean μ_i & variance σ_i^2 ($i = 1, 2, \dots, n$) then

$$\chi^2 = \left(\frac{X_i - \mu_i}{\sigma_i}\right)^2 \sim \chi^2_{n \text{ d.f.}}$$

Derivation of χ^2 Distribution :-

(Method of M.G.F.)

If X_i are indept. normal distribution with mean ~~the~~ μ_i & variance σ_i^2 ~~then~~ $(i = 1, 2, \ldots, n)$ then

$$\chi^2 = \sum_{i=1}^{n} \left(\frac{X_i - \mu_i}{\sigma_i} \right)^2 = \sum_{i=1}^{n} U_i^2 \; ; \text{ where } U_i = \frac{X_i - \mu_i}{\sigma_i}$$

Since X_i's are indept. U_i's are also indept.

$$M_{\chi^2}(t) = M_{\Sigma U_i^2}(t)$$

$$= \prod_{i=1}^{n} M_{U_i^2}(t)$$

$$= \left[M_{U_i^2}(t) \right]^n$$

Since $U_i \sim N(0,1)$ are identically distributed

$$M_{U_i^2}(t) = E\left[e^{tU_i^2} \right]$$

$$= \int_{-\infty}^{\infty} e^{tU_i^2} f(x_i)\, dx_i$$

$$= \int_{-\infty}^{\infty} e^{tU_i^2} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x_i - \mu_i}{\sigma_i}\right)^2} dx_i$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{tU_i^2} \frac{1}{\cancel{\sigma}} e^{-\frac{1}{2}U_i^2} \cancel{\sigma}\, dU_i$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{tU_i^2} e^{-\frac{1}{2}U_i^2} dU_i$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{(1-2t)}{2} U_i^2} dU_i$$

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$$= \frac{1}{\sqrt{2\pi}} \frac{\sqrt{\pi}}{\left(\frac{1-2t}{2}\right)^{1/2}}$$

$$\boxed{\text{as we know } \int_{-\infty}^{\infty} e^{-a^2x^2} dx = \frac{\sqrt{\pi}}{a}}$$

$$M_{U_i^2}(t) = \frac{1}{(1-2t)^{1/2}} = (1-2t)^{-1/2}$$

$$M_{\chi^2}(t) = \left[M_{U_i^2}(t)\right]^n = (1-2t)^{-n/2}$$

which is m.g.f. of Gamma variate with parameter $\frac{1}{2}$ & $\frac{n}{2}$.

Hence by uniqueness theorem of m.g.f.

$$\chi^2 = \sum_{i=1}^{n} \left(\frac{X_i - \mu_i}{\sigma_i}\right)^2 \text{ is } \gamma\left(\frac{1}{2}, \frac{n}{2}\right)$$

$$dP(\chi^2) = \frac{\left(\frac{1}{2}\right)^{n/2}}{\Gamma\frac{n}{2}} e^{-\chi^2/2} (\chi^2)^{\frac{n}{2}-1} d\chi^2$$

$$= \frac{1}{2^{n/2}\Gamma\frac{n}{2}} e^{-\chi^2/2} (\chi^2)^{\frac{n}{2}-1} d\chi^2 \quad ; 0 \le \chi^2 < \infty$$

Result

$$\frac{e^{-x} x^{n-1}}{\Gamma n} = \gamma(n)$$

or

$$\frac{e^{-ax} x^{n-1} (a)^n}{\Gamma n} = \gamma(a, n)$$

is required p.d.f. of χ^2-distn. with n degrees of freedom.

Remarks :-

1) If X is a r.v. & has a chi-square distribution with n degrees of freedom we write $X \sim \chi^2_{(n)}$ & its p.d.f. is given by

$$f(x) = \frac{e^{-x/2} (x)^{n/2 - 1}}{2^{n/2} \Gamma\frac{n}{2}} \quad ; \quad 0 \le x < \infty$$

(2) If $X \sim \chi^2_{(n)}$, then $\frac{X}{2} \sim \gamma\left(\frac{n}{2}\right)$
The p.d.f. of

$y = \frac{x}{2}$ is given by

$$g(y) = f(x) \left|\frac{dx}{dy}\right|$$

$$= \frac{1}{2^{n/2} \Gamma\frac{n}{2}} e^{-y} (2y)^{\frac{n}{2}-1} \cdot 2\, dy$$

$$= \frac{e^{-y} y^{n/2 - 1}}{\Gamma\frac{n}{2}} dy \quad ; \quad 0 \le y < \infty$$

which is p.d.f. of gamma variate.

Moment Generating Function of χ^2- distⁿ

Let $X \sim \chi^2_{(n)}$ then

$$M_X(t) = E(e^{tX}) = \int_0^{\infty} e^{tx} f(x)\, dx$$

$$= \int_0^{\infty} e^{tx} \cdot \frac{1}{2^{n/2} \Gamma\frac{n}{2}} e^{-x/2} x^{n/2 - 1} dx$$

$$= \frac{1}{2^{n/2} \Gamma\frac{n}{2}} \int_0^{\infty} e^{-\left(\frac{1-2t}{2}\right)x} x^{\frac{n}{2}-1} dx$$

Result

$$\int_0^{\infty} e^{-x} x^{n-1} dx = \Gamma n$$

$$\int_0^{\infty} e^{-ax} x^{n-1} dx = \frac{\Gamma n}{a^n}$$

$$= \frac{1}{2^{n/2} \Gamma n/2} \cdot \frac{\Gamma n/2}{\left(\frac{1-2t}{2}\right)^{n/2}} = \frac{1}{(1-2t)^{n/2}} = (1-2t)^{-n/2}$$

$\Rightarrow |2t| < 1$.

which is m.g.f. of χ^2- variate with n d.f.

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Cumulant Generating Function of χ^2-dist^{n} :-

If $X \sim \chi^2_{(n)}$ then

$$K_X(t) = \log M_X(t)$$
$$= \log (1-2t)^{-n/2}$$
$$= -\frac{n}{2} \log (1-2t)$$

Result

$\log(1-x) = \left[-x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots\right]$

$\log(1+x) = \left[x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} \cdots\cdots\right]$.

$$= -\frac{n}{2}\left[-2t - \frac{(2t)^2}{2} - \frac{(2t)^3}{3} - \frac{(2t)^4}{4} - \cdots\right]$$

$$K_X(t) = \frac{n}{2}\left[2t + \frac{(2t)^2}{2} + \frac{(2t)^3}{3} + \frac{(2t)^4}{4} + \cdots\right]$$

K_r are coefficient of $\frac{t^r}{r!}$ in $K_X(t)$

K_1 = coefficient of $\frac{t}{1!}$ in $K_X(t) = n$ = mean

K_2 = " " $\frac{t^2}{2!}$ in $K_X(t) = 2n$ = Variance

K_3 = " " $\frac{t^3}{3!}$ in $K_X(t) = 8n = \mu_3$

K_4 = " " $\frac{t^4}{4!}$ in $K_X(t) = 48n$

& $\mu_4 = K_4 + 3K_2^2 = 48n + 12n^2$

Coefficient of Skewness $\beta_1 = \frac{\mu_3^2}{\mu_2^3} = \frac{8}{n}$

Derivative of skewness $\gamma_1 = \pm\sqrt{\beta_1} = \sqrt{\frac{8}{n}}$

Coefficient of kurtosis $\beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{12}{n} + 3$

& Derivative of kurtosis $\gamma_2 = \beta_2 - 3 = \frac{12}{n}$.

Limiting form of χ^2- dist^{n} for large degree of freedom:-

If $X \sim \chi^2_{(n)}$ then $M_X(t) = (1-2t)^{-n/2}$; $|t| < \frac{1}{2}$

The m.g.f. of standard χ^2-variate Z is given by

$$M_{\frac{X-\mu}{\sigma}}(t) = e^{-\frac{\mu t}{\sigma}} M_X(t/\sigma)$$

$$= e^{-\frac{\mu t}{\sigma}} \left(1 - \frac{2t}{\sigma}\right)^{-n/2}$$

$$M_Z(t) = e^{-\frac{nt}{\sqrt{2n}}} \left(1 - \frac{2t}{\sqrt{2n}}\right)^{-n/2}$$

as mean $\mu = n$

variance $\sigma^2 = 2n$

$\sigma = \sqrt{2n}$

$$K_Z(t) = \log M_Z(t)$$

$$= -\frac{nt}{\sqrt{2n}} - \frac{n}{2} \log\left(1 - \frac{2t}{\sqrt{2n}}\right)$$

$$= -\sqrt{\frac{n}{2}}\, t - \frac{n}{2}\left[-\sqrt{\frac{2}{n}}\, t - \frac{2}{n}\frac{t^2}{2} - \left(\frac{2}{n}\right)^{3/2}\frac{t^3}{3} - \left(\frac{2}{n}\right)^2 \frac{t^4}{4} - \cdots\right]$$

$$= -\cancel{\sqrt{\frac{n}{2}}\, t} + \cancel{\sqrt{\frac{n}{2}}\, t} + \frac{t^2}{2} + \left(\frac{2}{n}\right)^{1/2}\frac{t^3}{3} + \cdots$$

$$= \frac{t^2}{2} + o(n^{-1/2})$$

where $o(n^{-1/2})$ are terms containing $n^{-1/2}$ & higher power of n in denominator.

$$\therefore \lim_{n \to \infty} K_Z(t) = \frac{t^2}{2}$$

$$\Rightarrow M_Z(t) = e^{t^2/2} \quad \text{as } n \to \infty$$

which is m.g.f. of standard normal variate

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Hence by uniqueness theorem of m.g.f., Z is asymptotically normal. In otherwords standard χ^2 variate tends to standard normal variate as $n \to \infty$. Thus χ^2 distribution tends to normal dist^{n} for large d.f. i.e.

$n > 30$; χ^2 approximate to normal dist^{n}.

$n = 30$; we use χ^2 dist^{n}.

Characteristic Function of χ^2 - dist^{n} :-

If $X \sim \chi^2_{(n)}$ then

$$\phi_X(t) = E(e^{itx}) = \int_0^\infty e^{itx} f(x)\, dx$$

$$= \int_0^\infty e^{itx} \frac{e^{-x/2} x^{n/2-1}}{2^{n/2} \Gamma\frac{n}{2}} dx$$

$$= \frac{\int_0^\infty e^{-\frac{(1-2it)}{2}x} x^{n/2-1} dx}{2^{n/2} \Gamma n/2}$$

$$\phi_X(t) = \frac{\Gamma\frac{n}{2}}{\left(\frac{1-2it}{2}\right)^{n/2}} \times \frac{1}{2^{n/2}\Gamma\frac{n}{2}} = (1-2it)^{-n/2}$$

Mode & Skewness of χ^2 - dist^{n} :-

Let $X \sim \chi^2_{(n)}$ so that

$$f(x) = \frac{1}{2^{n/2}\Gamma n/2} e^{-x/2} x^{n/2-1} dx.$$

mode of the dist^{n} is the solution of

$f'(x) = 0$ & $f''(x) < 0$

$$\therefore f'(x) = \frac{1}{2^{n/2}\,\Gamma n/2}\left\{-\frac{1}{2}e^{-x/2}x^{n/2-1} + e^{-x/2}x^{\frac{n}{2}-2}\left(\frac{n}{2}-1\right)\right\}$$

$$= 0$$

$$\Rightarrow -\frac{1}{2} + \left(\frac{n}{2}-1\right)\frac{1}{x} = 0$$

$$\Rightarrow \frac{x}{2} = \frac{n}{2} - 1 \Rightarrow x = n-2$$

at pt. $x = n-2$, $f''(x) < 0$

$$f''(x) = \frac{1}{2^{n/2}\,\Gamma\frac{n}{2}}\left\{\frac{1}{4}e^{-x/2}x^{\frac{n}{2}-1} - \frac{1}{2}e^{-x/2}\left(\frac{n}{2}-1\right)x^{\frac{n}{2}-2}\right.$$

$$+\left(\frac{n}{2}-1\right)\left(-\frac{1}{2}\right)e^{-x/2}x^{\frac{n}{2}-2} + \left(\frac{n}{2}-1\right)\left(\frac{n}{2}-2\right)$$

$$\left. e^{-x/2}x^{\frac{n}{2}-3}\right\}$$

putting $x = n-2$ we get

$f''(x) < 0$

Hence mode of the chi-square dist^{n} with n d.f. is $(n-2)$.

Karl Pearson coefficient of skewness is given by

$$\text{skewness} = \frac{\text{mean} - \text{mode}}{\text{S.D.}} = \frac{n-(n-2)}{\sqrt{2n}} = \sqrt{\frac{2}{n}}$$

$\sqrt{\frac{2}{n}}$ is ≥ 0 for $n \geq 1$

The χ^2 – dist^{n} is positively skewed distribution.

Additive property of χ^2 dist^{n} :—

Statement :— The sum of indept. χ^2-variate is also a χ^2-variate. More precisely if X_i $(i=1,2,\dots,k)$ are indept. χ^2-variate with n_i d.f. then the sum $\sum_{i=1}^{k} x_i$ is also a chi-square variate with $\sum_{i=1}^{k} n_i$ d.f.

Proof :— We have m.g.f. of χ^2-dist^{n} as follows

$$M_{X_i}(t) = (1-2t)^{-n/2} \quad ; \quad i=1,2,\dots,k$$

The m.g.f. of the sum $\sum_{i=1}^{k} X_i$ is given by

$$M_{\sum X_i}(t) = \prod_{i=1}^{n} M_{X_i}(t)$$

$$= M_{X_1}(t) \cdot M_{X_2}(t) \cdots M_{X_k}(t)$$

[$\because$ X_i's are indept.]

$$= (1-2t)^{-n_1/2} (1-2t)^{-n_2/2} \cdots (1-2t)^{-n_k/2}$$

$$= (1-2t)^{-(n_1+n_2+\dots+n_k)/2}$$

Which is m.g.f. of a χ^2-variate with $(n_1+n_2+\dots+n_k)$ d.f.. Hence by uniqueness theorem of m.g.f.'s $\sum_{i=1}^{k} X_i$ is a χ^2-variate with $\sum_{i=1}^{k} n_i$ d.f.

Remarks :—

1) Converse of the above property of χ^2 is also true i.e. if X_i ; $i=1,2,\dots,k$ are χ^2-variate with n_i ; $i=1,2,\dots,k$ d.f. respectively and if $\sum_{i=1}^{k} X_i$ is a χ^2-variate with $\sum_{i=1}^{k} n_i$ d.f., then X_i's are independent.

2) If X and Y are independent non negative variate such that X+Y follows chi-square distribution with n_1+n_2 d.f. and if one of them say X is a

χ^2-variate with n_1 d.f then the other viz Y is a χ^2- variate with n_2 d.f.

Proof:- Since X and Y indept variates, we have

$$M_{X+Y}(t) = M_X(t) \cdot M_Y(t)$$

$$(1-2t)^{-n_1+n_2/2} = (1-2t)^{\frac{-(n_1)}{2}} \cdot M_Y(t)$$

$[\because X+Y \sim \chi^2_{(n_1+n_2)}$ and $X \sim \chi^2_{(n_1)}]$

$$M_Y(t) = (1-2t)^{-n_2/2}$$

Which is m.g.f of χ^2- variate with n_2 d.f. Hence by uniqueness thm. of m.g.f's $Y \sim \chi^2_{(n_2)}$.

3) Let $x_1, x_2, \ldots, x_n$ be independently distributed as standard normal variate i.e. $N(0,1)$ then

$$\sum_{i=1}^{n} X_i^2 = Q_1 + Q_2 + \ldots + Q_k$$

Where each Q_i is a sum of squares of lineare combination of $X_1, X_2, \ldots, X_n$ with n_i degrees of freedom. Then $n_1+n_2+\ldots+n_k = n$, the quantities $Q_1, Q_2, \ldots, Q_k$ are indepent χ^2- variate with $n_1, n_2, \ldots, n_k$ d.f. respectively.

Chi-square probability curve:- Let $X \sim \chi^2_{(n)}$.

As $$f(x) = \frac{1}{2^{n/2}\,\Gamma n/2}\, e^{-x/2}\, x^{(n/2)-1} \quad ; 0 \le x < \infty$$

$$f'(x) = \left[\frac{n-2-x}{2x}\right] f(x)$$

since $x > 0$ and $f(x)$ being p.d.f. is always non-negative, we get.

$f'(x) \le 0$, if $(n-2) \le 0$

for all values of x. Thus the χ^2- prob. curve for 1 and 2 d.f. is monotonically decreasing.

when

$n > 2$

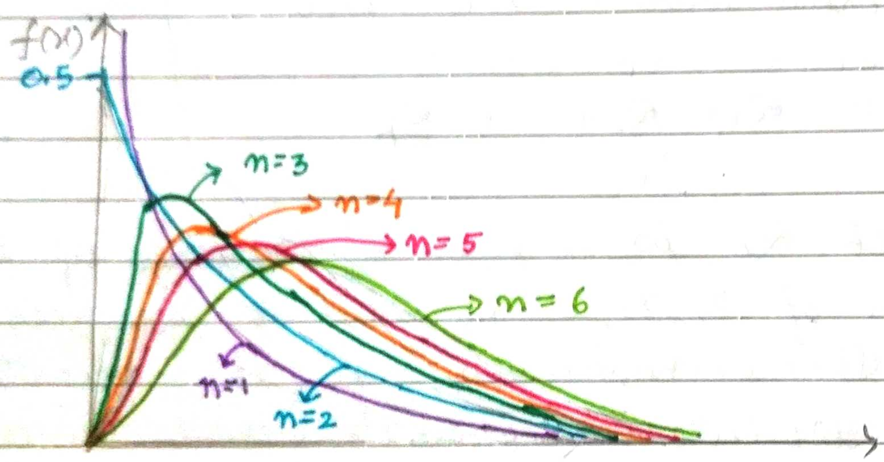
$$f'(x) = \begin{cases} > 0, & \text{if } x < (n-2) \\ = 0, & \text{if } x = (n-2) \\ < 0, & \text{if } x > (n-2) \end{cases}$$

$\Rightarrow$ $f(x)$ is monotonically increasing for $n > (2)$

i.e. $0 < x < (n-2)$

& $f(x)$ is monotonically decreasing for $(n-2) < x < \infty$

& at $x = (n-2)$ it attains the maximum value.



Theorem①:- If χ_1^2 and χ_2^2 are two independent χ^2-variate with n_1 and n_2 d.f. respectively, then

$\frac{\chi_1^2}{\chi_2^2}$ is a $\beta_2\left(\frac{n_1}{2}, \frac{n_2}{2}\right)$ variate.

Proof:-

Since χ_1^2 and χ_2^2 are indept. χ^2-variate with n_1 and n_2 d.f. respectively, their joint probability differential is given by the compound probability theorem as

$$dP(\chi_1^2, \chi_2^2) = dP_1(\chi_1^2)\, dP_2(\chi_2^2)$$

$$= \left[\frac{1}{2^{n_1/2}\,\Gamma\frac{n_1}{2}} e^{-\chi_1^2/2} (\chi_1^2)^{\frac{n_1}{2}-1} d\chi_1^2\right]$$

$$\times \left[\frac{1}{2^{n_2/2}\,\Gamma\frac{n_2}{2}} e^{-\chi_2^2/2} (\chi_2^2)^{\frac{n_2}{2}-1} d\chi_2^2\right]$$

$$= \frac{1}{2^{\frac{n_1+n_2}{2}}\,\Gamma\frac{n_1}{2}\,\Gamma\frac{n_2}{2}} e^{-\frac{\chi_1^2+\chi_2^2}{2}} (\chi_1^2)^{\frac{n_1}{2}-1} (\chi_2^2)^{\frac{n_2}{2}-1} d\chi_1^2\, d\chi_2^2$$

Let us make the transformation:

$$u = \chi_1^2/\chi_2^2 \quad \& \quad v = \chi_2^2$$

So that $\chi_1^2 = uv$

& $\chi_2^2 = v$

Now jacobian of transformation is given by.

$$|J| = \frac{\partial(\chi_1^2, \chi_2^2)}{\partial(u,v)} = \begin{vmatrix} v & u \\ 0 & 1 \end{vmatrix} = v$$

Thus the joint distn. of r.v.'s U and V becomes

$$dG(u,v) = \frac{1}{2^{\frac{n_1+n_2}{2}}\,\Gamma\frac{n_1}{2}\,\Gamma\frac{n_2}{2}} e^{-(1+u)\frac{v}{2}} (uv)^{\frac{n_1}{2}-1} (v)^{\frac{n_2}{2}-1} \times |J| \times du\, dv$$

$$= \frac{1}{2^{\frac{n_1+n_2}{2}}\,\Gamma\frac{n_1}{2}\,\Gamma\frac{n_2}{2}} e^{-(1+u)\frac{v}{2}} (u)^{\frac{n_1}{2}-1} (v)^{\frac{n_1+n_2}{2}-2}\, v\, du\, dv$$

$$= \frac{1}{2^{\frac{n_1+n_2}{2}}\,\Gamma\frac{n_1}{2}\,\Gamma\frac{n_2}{2}} e^{-(1+u)\frac{v}{2}} u^{\frac{n_1}{2}-1} (v)^{\frac{n_1+n_2}{2}-1} du\, dv$$

$$0 \le (u,v) < \infty$$

We want the marginal distn. of u so integrating w.r.t. v over the range 0 to ∞, we get the marginal distn. of U

$$dP_1(u) = \int_0^\infty dP(u,v)$$

$$= \int_0^\infty \left(\frac{1}{2^{\frac{n_1+n_2}{2}} \Gamma\frac{n_1}{2} \Gamma\frac{n_2}{2}} u^{\frac{n_1}{2}-1} du\right)\left(e^{-(1+u)\frac{v}{2}} (v)^{\frac{n_1+n_2}{2}-1} dv\right)$$

$$= \frac{u^{\frac{n_1}{2}-1} du}{2^{\frac{n_1+n_2}{2}} \Gamma\frac{n_1}{2} \Gamma\frac{n_2}{2}} \int_0^\infty e^{-(1+u)\frac{v}{2}} (v)^{\frac{n_1+n_2}{2}-1} dv$$

As

$$\int_0^\infty e^{-x} x^{n-1} dx = \Gamma n$$

$$\int_0^\infty e^{-ax} x^{n-1} dx = \frac{\Gamma n}{a^n}$$

$$B\left(\frac{n_1}{2}, \frac{n_2}{2}\right) = \frac{\Gamma\frac{n_1}{2} \Gamma\frac{n_2}{2}}{\Gamma\frac{n_1+n_2}{2}}$$

$$= \frac{u^{\frac{n_1}{2}-1} du}{2^{\frac{n_1+n_2}{2}} \Gamma\frac{n_1}{2} \Gamma\frac{n_2}{2}} \cdot \frac{\Gamma\frac{n_1+n_2}{2}}{\left(\frac{1+u}{2}\right)^{\frac{n_1+n_2}{2}}}$$

$$= \frac{u^{\frac{n_1}{2}-1}}{(1+u)^{\frac{n_1+n_2}{2}}} \frac{du}{B\left(\frac{n_1}{2}, \frac{n_2}{2}\right)} \qquad 0 \le u < \infty$$

is p.d.f. of $\beta_2\left(\frac{n_1}{2}, \frac{n_2}{2}\right)$

Hence $U = \frac{\chi_1^2}{\chi_2^2}$ is a $\beta_2\left(\frac{n_1}{2}, \frac{n_2}{2}\right)$ variate

Theorem (2):— If χ_1^2 and χ_2^2 are independent χ^2-variates with n_1 and n_2 d.f respectively then $U = \frac{\chi_1^2}{\chi_1^2+\chi_2^2}$ and $V = \chi_1^2+\chi_2^2$ are indeptly. distributed U as $\beta_1\left(\frac{n_1}{2}, \frac{n_2}{2}\right)$ variate and V as a χ^2-variate with (n_1+n_2) d.f.

Proof:

Since χ_1^2 and χ_2^2 are indept. χ^2-variate with n_1 and n_2 d.f. respectively their joint prob. differential is given by the compound prob. thm. as

$$dP(\chi_1^2, \chi_2^2) = dP_1(\chi_1^2)\, dP_2(\chi_2^2)$$

$$dP(\chi_1^2, \chi_2^2) = \frac{1}{2^{\frac{n_1+n_2}{2}} \Gamma\frac{n_1}{2} \Gamma\frac{n_2}{2}} e^{-(\chi_1^2+\chi_2^2)/2} (\chi_1^2)^{\frac{n_1}{2}-1} (\chi_2^2)^{\frac{n_2}{2}-1} d\chi_1^2 \, d\chi_2^2$$

$0 \le (\chi_1^2, \chi_2^2) < \infty$

Let us transform to u and v defined as follows:

$$u = \frac{\chi_1^2}{\chi_1^2+\chi_2^2} \quad \text{and} \quad v = \chi_1^2 + \chi_2^2$$

So $\chi_1^2 = uv$

& $\chi_2^2 = (1-u)v$

the range of χ_1^2 and χ_2^2 are 0 ~~to~~ ∞

u ranges from 0 to 1 and v from 0 to ∞

Jacobian of transformation J is

$$J = \begin{vmatrix} \cancel{v} & u \\ -v & (1-u) \end{vmatrix} = v$$

$$dP(u,v) = \frac{1}{2^{(n_1+n_2)/2} \Gamma\frac{n_1}{2} \Gamma\frac{n_2}{2}} e^{-v/2} (uv)^{\frac{n_1}{2}-1} \{(1-u)v\}^{\frac{n_2}{2}-1} du\,dv \times |J|$$

$$= \frac{1}{2^{\frac{n_1+n_2}{2}} \Gamma\frac{n_1}{2} \Gamma\frac{n_2}{2}} u^{\frac{n_1}{2}-1} (1-u)^{\frac{n_2}{2}-1} e^{-\frac{v}{2}} (v)^{\frac{n_1+n_2}{2}-2} \times v \, du\,dv$$

$$= \frac{1}{2^{\frac{n_1+n_2}{2}} \Gamma\frac{n_1}{2} \Gamma\frac{n_2}{2}} u^{\frac{n_1}{2}-1} (1-u)^{\frac{n_2}{2}-1} du \; e^{-\frac{v}{2}} (v)^{\frac{n_1+n_2}{2}-1} dv$$

$$= \left[\frac{\Gamma\frac{n_1+n_2}{2}}{\Gamma\frac{n_1}{2} \Gamma\frac{n_2}{2}} u^{\frac{n_1}{2}-1} (1-u)^{\frac{n_2}{2}-1} du \right]$$

$$\times \left[\frac{1}{2^{\frac{n_1+n_2}{2}} \Gamma\frac{n_1+n_2}{2}} e^{-v/2} (v)^{\frac{n_1+n_2}{2}-1} dv \right]$$

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Since the joint prob. differential of U and V is product of their respective prob. differential. U and V are indeptly. distributed with

$$dG_1(u) = \frac{1}{B\left(\frac{n_1}{2}, \frac{n_2}{2}\right)} u^{\frac{n_1}{2}-1} (1-u)^{\frac{n_2}{2}-1} du \qquad 0 \le u < 1$$

$$\& \quad dG_2(v) = \frac{1}{2^{\frac{n_1+n_2}{2}} \Gamma\frac{n_1+n_2}{2}} e^{-\frac{v}{2}} (v)^{\frac{n_1+n_2}{2}-1} dv \qquad 0 \le v < \infty$$

i.e. $U \sim \beta_1\left(\frac{n_1}{2}, \frac{n_2}{2}\right)$ variate &

$V \sim \chi^2$ variate (n_1+n_2) d.f.

Remark :— If $X \sim \chi^2_{n_1}$ & $Y \sim \chi^2_{n_2}$ are χ^2 indept. χ^2-variates then

① $X+Y \sim \chi^2_{(n_1+n_2)}$

② $\frac{X}{Y} \sim \beta_2\left(\frac{n_1}{2}, \frac{n_2}{2}\right)$.

③ $\frac{X}{X+Y} \sim \beta_1\left(\frac{n_1}{2}, \frac{n_2}{2}\right)$