Recursion - In a sequence, the next term is determined using the previous term.

ex Sequence 3, 8, 13, ...

$$a_n = a_{n-1} + 5 \; ; \; a_1 = 3, \; 2 \leq n < \infty$$

Recurrence Relation - Based on Recursion.

A Recurrence Relation actually defines a sequence by establishing the rth value in terms of its predecessors (previous terms). It is also called **Difference Equation**.

Linear Recurrence Relation -

$$C_0(r)a_r + C_1(r)a_{r-1} + \cdots + C_k(r)a_{r-k} = f(r)$$

with $r \geq k$ where $C_0(r), C_1(r), \ldots, C_k(r)$ are not 0 then it is **Linear Recurrence Relation with Constant coefficients**.

If $f(r) = 0$ then the Recurrence Relation is called **Homogeneous Recurrence Relation**, else **Non-homogeneous Recurrence Relation**.

ex. $a_r - 3a_{r-1} + 2a_{r-2} = r^2 + 1$ ——(1)

(1) is a Non homogeneous Linear Recurrence Relation of order 2 (order 2 ∵ of 2 in $(r-2)$ in $2a_{r-2}$)

$$(a_r)^2 + (a_{r-1})^2 + 1 = 0 \quad \text{——} \quad (2)$$

(2) is a Homogeneous Non Linear Recurrence Relation of order 1. (order 1 due to 1 in $(r-1)$ in a_{r-1})

Non linear $\because$ of $(a_r)^2$.

<u>Fibonacci Relation</u> - $a_r = a_{r-1} + a_{r-2}$, $r \geq 2$ —— (1)

initial conditions: $a_0 = a_1 = 1$.

The sequence $(a_0, a_1, a_2, a_3, \ldots, a_r, \ldots)$ of Fibonacci numbers is the <u>Fibonacci Sequence</u>.

$F(x) = \sum_{r=0}^{\infty} a_r x^r$ is called the <u>Generating function</u>

for the sequence $(a_0, a_1, a_2, \ldots)$

$$\therefore F(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_r x^r + \cdots$$

$$x F(x) = a_0 x + a_1 x^2 + a_2 x^3 + \cdots a_{r-1} x^r + \cdots$$

$$x^2 F(x) = a_0 x^2 + a_1 x^3 + \cdots + a_{r-2} x^r + \cdots$$

$$\therefore F(x) - x F(x) - x^2 F(x) = a_0 + (a_1 - a_0) x + (a_2 - a_1 - a_0) x^2$$
$$+ (a_3 - a_2 - a_1) x^3 + \cdots + (a_r - a_{r-1} - a_{r-2}) x^r + \cdots \quad \text{——} (1)$$

<u>Note</u>: Fibonacci term is $a_r = a_{r-1} + a_{r-2}$

$$\therefore a_2 = a_1 + a_0 = (a_2 - a_1 - a_0) x^2 = 0 \cdot x^2 = 0$$

So we get $F(x) = a_0 + (a_1 - a_0) x$

From (1), we get

or $(1 - x - x^2) F(x) = a_0 + (a_1 - a_0) x$

$$\therefore F(x) = \frac{a_0 + (a_1 - a_0) x}{1 - x - x^2}$$

The denominator is a Quadratic eqn

$$\therefore F(x) = \frac{a_0 + (a_1 - a_0)x}{\left(1 - \frac{1+\sqrt{5}}{2}x\right)\left(1 - \frac{1-\sqrt{5}}{2}x\right)}$$

$$= \frac{C_1}{1 - \frac{1+\sqrt{5}}{2}x} + \frac{C_2}{1 - \frac{1-\sqrt{5}}{2}x}$$ for every value of a_0 & a_1.

C_1, C_2 are to be determined

Now assuming, $a = \frac{1+\sqrt{5}}{2}$ & $b = \frac{1-\sqrt{5}}{2}$ we get

$$F(x) = \frac{C_1}{1-ax} + \frac{C_2}{1-bx}$$

$$= C_1 \sum_{r=0}^{\infty} a^r x^r + C_2 \sum_{r=0}^{\infty} b^r x^r$$

$$= \sum_{r=0}^{\infty} (C_1 a^r + C_2 b^r) x^r$$

But $F(x) = \sum_{r=0}^{\infty} a_r x^r$ $\therefore a_r = C_1 a^r + C_2 b^r$

$$= C_1 \left(\frac{1+\sqrt{5}}{2}\right)^r + C_2 \left(\frac{1-\sqrt{5}}{2}\right)^r \quad (2)$$ for every $r \geq 0$

Using initial conditions $a_0 = a_1 = 1$ ie, first put $r=0$ in (2) then $r=1$ in (2), we get

$$C_1 = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right) \text{ \& } C_2 = \frac{-1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)$$

$$\therefore a_r = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{r+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{r+1}\right]$$ Ans.

We have found the rth term ~~term~~ of the Fibonacci sequence ie, the rth Fibonacci no.