Composite function :-

If $u = f(x, y)$ be a function of two variables x and y where x and y are functions of variable t only given by

$$x = \phi(t), \quad y = \psi(t)$$

then u is called a Composite function of single independent variable t.

If x and y are functions of two variables t_1 & t_2 i.e.

$$x = \phi(t_1, t_2) \quad \text{and} \quad y = \psi(t_1, t_2)$$

then u is called composite function of two independent variables t_1 and t_2.

Total Differential Coefficient:

If $u = f(x, y)$, where $x = \phi(t)$, $y = \psi(t)$

then $\frac{du}{dt}$ is called total differential coefficient of u w.r.t t

Derivative of Composite functions:

1. If u is composite function of t given by relations $u = f(x, y)$, $x = \phi(t)$ and $y = F(t)$, where u possesses continuous partial derivatives and x, y possesses derivates then

$$\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt} \quad \text{(Chain Rule)}$$

2. Change of variables:

If $u = f(x, y)$ be function of x and y having Continuous partial derivatives w.r.t x & y

$x = \phi_1(t_1, t_2)$ and $y = \phi_2(t_1, t_2)$ having Continuous partial derivatives wrt t_1 & t_2 then

$$\frac{\partial u}{\partial t_1} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial t_1} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial t_1}$$

and $$\frac{\partial u}{\partial t_2} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial t_2} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial t_2}$$

Derivative of Implicit functions:

A function $f(x, y) = 0$ in two variables x and y is said to be an implicit function if it is defined by relation s.t. neither y can be expressed directly in terms of x nor x in terms of y

If $f(x, y) = c$, then $\frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y}$, provided $\frac{\partial f}{\partial y} \neq 0$

Proof $$\frac{df}{dx} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dx} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dx}$$

$$\Rightarrow \frac{df}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dx} \quad \text{—— (1)}$$

But $f(x, y) = c \Rightarrow \frac{df}{dx} = \frac{dc}{dx} = 0$ —(2)

from (1) & (2)

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y}, \frac{\partial f}{\partial y} \neq 0$$

Q.1 If $u = xy^2 + x^2 y$, $x = at^2$, $y = 2at$, then find $\frac{du}{dt}$

Sol. Since $\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt}$ — (1) (chain rule)

now $\frac{\partial u}{\partial x} = y^2 + 2xy$, $\frac{\partial u}{\partial y} = 2xy + x^2$

$\frac{dx}{dt} = 2at$ & $\frac{dy}{dt} = 2a$

from (1)

$$\frac{du}{dt} = (y^2 + 2xy)\, 2at + (2xy + x^2)\, 2a$$

$$= (4a^2t^2 + 4a^2t^3)\, 2at + (4a^2t^3 + a^2t^4)\, 2a$$

$$= 2a^3t^3(8 + 5t)$$

Q2 If $u = f(y-z, z-x, x-y)$, then prove that

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$$

Sol. Let $y - z = t_1$, $z - x = t_2$ and $x - y = t_3$

then $u = f(t_1, t_2, t_3)$ where t_1, t_2 & t_3 are function of x, y & z

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial t_1} \cdot \frac{\partial t_1}{\partial x} + \frac{\partial u}{\partial t_2} \cdot \frac{\partial t_2}{\partial x} + \frac{\partial u}{\partial t_3} \cdot \frac{\partial t_3}{\partial x}$$

$$= \frac{\partial u}{\partial t_1}(0) + \frac{\partial u}{\partial t_2}(-1) + \frac{\partial u}{\partial t_3}(1) = -\frac{\partial u}{\partial t_2} + \frac{\partial u}{\partial t_3} \quad \text{—(1)}$$

Similarly $\frac{\partial u}{\partial y} = \frac{\partial u}{\partial t_1}\cdot\frac{\partial t_1}{\partial y} + \frac{\partial u}{\partial t_2}\cdot\frac{\partial t_2}{\partial y} + \frac{\partial u}{\partial t_3}\cdot\frac{\partial t_3}{\partial y}$

$$= \frac{\partial u}{\partial t_1} - \frac{\partial u}{\partial t_3} \quad \text{---} \quad (2)$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial t_1}\cdot\frac{\partial t_1}{\partial z} + \frac{\partial u}{\partial t_2}\cdot\frac{\partial t_2}{\partial z} + \frac{\partial u}{\partial t_3}\cdot\frac{\partial t_3}{\partial z}$$

$$= -\frac{\partial u}{\partial t_1} + \frac{\partial u}{\partial t_2} \quad \text{--} \quad (3)$$

On adding (1), (2) & (3), we get

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$$

Q3 If $x^y + y^x = c$, find $\frac{dy}{dx}$

Sol. Let $f(x, y) = x^y + y^x$, then $f(x, y) = c$

So $\frac{dy}{dx} = \frac{-\partial f/\partial x}{\partial f/\partial y}$

But $\frac{\partial f}{\partial x} = yx^{y-1} + y^x \log y$ & $\frac{\partial f}{\partial y} = x^y \log x + xy^{x-1}$

$$\frac{dy}{dx} = -\frac{yx^{y-1} + y^x \log y}{x^y \log x + xy^{x-1}}$$

Q4 If $x = r\cos\theta$, $y = r\sin\theta$ then prove that

(i) $\left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2 = 1$

(ii) $\frac{\partial r}{\partial x} = \frac{\partial x}{\partial r}$

(iii) $\frac{\partial x}{\partial \theta} = r^2 \frac{\partial \theta}{\partial x}$

(iv) $\frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2} = \frac{1}{r}$

Sol. (i) $x = r\cos\theta$, $y = r\sin\theta$ -- (1)

Squaring & adding these equations

$x^2 + y^2 = r^2$ also $\tan\theta = \frac{y}{x} \Rightarrow \theta = \tan^{-1}(y/x)$

from (1) $\frac{\partial x}{\partial r} = \cos\theta$ & $\frac{\partial y}{\partial r} = \sin\theta$

from $x^2 + y^2 = r^2$

diff. partially wrt x

$2x = 2r\frac{\partial r}{\partial x} \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r} = \frac{r\cos\theta}{r} = \cos\theta$ — (3)

also $\frac{\partial r}{\partial y} = \frac{y}{r} = \frac{r\sin\theta}{r} = \sin\theta$ — (4)

Squaring & adding (3) & (4)

$$\left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2 = 1$$

(ii) $\frac{\partial r}{\partial x} = \frac{x}{r} = \cos\theta$ — (4) and $\frac{\partial x}{\partial r} = \cos\theta$ — (5)

from (4) & (5)

so $\frac{\partial r}{\partial x} = \frac{\partial x}{\partial r}$

(iii) $x = r\cos\theta$, $\frac{\partial x}{\partial \theta} = -r\sin\theta$ --- (6)

$\theta = \tan^{-1}(y/x)$,

$$\frac{\partial \theta}{\partial x} = \frac{1}{1+y^2/x^2} \cdot \frac{-y}{x^2} = -\frac{y}{x^2+y^2} = -\frac{r\sin\theta}{r^2}$$

$$= -\frac{\sin\theta}{r} \quad \text{--- (7)}$$

from (6) & (7)

$$\frac{\partial x}{\partial \theta} = r^2 \frac{\partial \theta}{\partial x}$$

(iv) $\frac{\partial r}{\partial x} = \frac{x}{r}$

diff. partially wrt x

$$\frac{\partial^2 r}{\partial x^2} = \frac{1}{r} - \frac{x}{r^2}\frac{\partial r}{\partial x} = \frac{1}{r} - \frac{x}{r^2}\cdot\frac{x}{r}$$

$$= \frac{1}{r} - \frac{x^2}{r^3} \quad \text{--- (8)}$$

also $\frac{\partial r}{\partial y} = \frac{y}{r}$

diff. partially w.r.t. y

$$\frac{\partial^2 r}{\partial y^2} = \frac{1}{r} - \frac{y}{r^2}\frac{\partial r}{\partial y} = \frac{1}{r} - \frac{y}{r^2}\cdot\frac{y}{r}$$

$$= \frac{1}{r} - \frac{y^2}{r^3} \quad \text{--- (9)}$$

Adding (8) & (9)

$$\frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2} = \frac{2}{r} - \frac{1}{r^3}(x^2+y^2) = \frac{2}{r} - \frac{r^2}{r^3}$$

$$= \frac{1}{r}$$