Theorem 4. If K & T are subsets of vector space $V(F)$, then

(a) $K \subset T \Rightarrow L(K) \subset L(T)$

(b) $K \subset L(T) \Rightarrow L(K) \subset L(T)$

(c) K is subspace of $V(F) \Leftrightarrow L(K) = K$

(d) $L\{K \cup T\} = L(K) + L(T)$

(e) $L\{L(K)\} = L(K)$

Proof:- (a) given $K \subset T$

let $\alpha \in L(K) \Rightarrow \alpha = \sum_{i=1}^{n} a_i \alpha_i$ — (1) where $a_i \in F$

& $\alpha_i \in K$ $(i = 1, 2, \dots n)$

$(i = 1, 2, \dots n)$

$\because \alpha_i \in K$ & $K \subset T$ $(i = 1, 2, \dots n)$

$\Rightarrow \alpha_i \in T$ $(i = 1, 2, \dots n)$

$\Rightarrow \sum_{i=1}^{n} a_i \alpha_i \in L(T)$ ($\sum a_i \alpha_i$ is LC)

$\Rightarrow \alpha \in L(T)$ (by (1))

Hence $\alpha \in L(K) \Rightarrow \alpha \in L(T)$

$\Rightarrow L(K) \subset L(T)$

(b) let $\alpha \in L(K) \Rightarrow \alpha = \sum_{i=1}^{n} a_i \alpha_i$ where

$a_i \in F$ & $\alpha_i \in K$ $(i = 1, 2, \cdots n)$

$\because \alpha_i \in K$ & $K \subset L(T)$ (given)

$\Rightarrow \alpha_i \in L(T)$ $(i = 1, 2, \cdots n)$

$\Rightarrow \sum_{i=1}^{n} a_i \alpha_i \in L(T)$ ($\because$ $L(T)$ is subspace)

Hence $\alpha \in L(K) \Rightarrow \alpha \in L(T)$

$\Rightarrow L(K) \subset L(T)$

(c) Since K is subspace of $V(F)$.

Then we prove that $L(K) = K$

$\{A = B \Leftrightarrow A \subset B$ & $B \subset A\}$

let $\alpha \in L(K) \Rightarrow \alpha = \sum_{i=1}^{n} a_i \alpha_i$ —(i) where $a_i \in F$

& $\alpha_i \in K$ $(i = 1, 2, \cdots n)$

Since K is subspace of $V(F)$

$\Rightarrow$ K is closed w.r. to scalar multipli-

-cation and vector addition

$a_i \in F$, $\alpha_i \in K \Rightarrow a_i \alpha_i \in K$; $i = 1, 2, \cdots n$

$\Rightarrow \sum_{i=1}^{n} a_i \alpha_i \in K$

$\Rightarrow \alpha \in K$ (by (i))

$\Rightarrow L(K) \subset K$ — (ii)

Now by theorem 1, we know that $L(K)$ is the smallest subspace containing K

$K \subset L(K)$ — (iii)

By (ii) & (iii) $\Rightarrow K = L(K)$

(d) let $\alpha \in L(K \cup T)$

$\Rightarrow \alpha = \sum_{i=1}^{n} a_i \alpha_i$ — (i) where $a_i \in F$

& $\alpha_i \in K \cup T$; $i = 1, 2, \cdots n$

$\Rightarrow \alpha = \sum_{i=1}^{n} a_i \alpha_i = \sum_{i=1}^{m} a_i \alpha_i + \sum_{i=m+1}^{n} a_i \alpha_i$ — (ii)

where $\alpha_i (i = 1, 2, \cdots m) \in K$ &

$\alpha_i (i = m+1, m+2, \cdots n) \in T$

{ since $\alpha_i \in K \cup T$ $(i = 1, 2, \cdots n)$
$\Rightarrow \alpha_i$ is either an Element of K or T or both, so we divide α_i in such a way }

by (i) & (ii)

$\alpha \in L(K \cup T) \Rightarrow \alpha \in L(K) + L(T)$

$\Rightarrow L(K \cup T) \subset L(K) + L(T)$ — (iii)

let $\beta = \gamma + \delta \in L(K) + L(T)$ — (iv)

$\Rightarrow \gamma \in L(K), \delta \in L(T)$

$\Rightarrow \gamma = \sum_{i=1}^{m} a_i \alpha_i, \delta = \sum_{i=m+1}^{n} a_i \alpha_i$

where $a_i \in F$ $(i=1,2,\dots n)$ &

α_i $(i=1,2,\dots m) \in K$, α_i $(i=m+1,\dots n) \in T$

$$\Rightarrow \beta = \sum_{i=1}^{m} a_i \alpha_i + \sum_{i=m+1}^{n} a_i \alpha_i$$

$\Rightarrow$ β is LC of elements of $K \cup T$

$\Rightarrow$ $\beta \in L(K \cup T)$ — (V)

By (iv) & (V) $\Rightarrow$ $L(K) + L(T) \subset L(K \cup T)$ — (vi)

By (iii) & (vi) $\Rightarrow$ $L(K) + L(T) = L(K \cup T)$

(2) from (1), if K is subspace of $V(F)$

then $L(K) = K$

Since $L(K)$ is subspace of $V(F)$

$\{K \rightarrow L(K)\}$

$L\{L(K)\} = L(K)$

H. P.