Linear Independence (L.I.) :-

Let V(F) be a vector space then the vectors $\alpha_1, \alpha_2, \cdots, \alpha_n$ are called L.I. if ∃ scalars $a_1, a_2, \cdots, a_n \in F$ s.t.

$$a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n = 0 \Rightarrow a_i = 0 \ \forall i = 1, 2, \cdots n$$

or

The L.C. of non zero vectors is Equal to zero if each & every scalar coffecient is zero.

For Eg:- In vector space C(R),

let $\alpha_1, \alpha_2 \in C$ & $a_1\alpha_1 + a_2\alpha_2 = 0$ where $a_1, a_2 \in R$

if the vectors α_1, α_2 are L.I.

then $a_1 = 0$ & $a_2 = 0$

but in the case C(C) when $\alpha_1 = 1$, $\alpha_2 = i$

then $a_1\alpha_1 + a_2\alpha_2 = 0 \Rightarrow a_1 \cdot 1 + a_2 \cdot i = 0$

$$\Rightarrow a_1 + a_2 i = 0$$

$$\Rightarrow a_2 = -\frac{1}{i} a_1$$

For Eg. if $a_1 = i$, $a_2 = -1$

$\Rightarrow i(1) + (-1)i = 0$ but $a_1 \neq 0$, $a_2 \neq 0$

$\Rightarrow \alpha_1, \alpha_2$ are not L.I.

Linear Dependent :- (L.D.) Let V(F) be a vector space then the vectors $\alpha_1, \alpha_2, \cdots, \alpha_n$ are called L.D. if ∃ scalars $a_1, a_2, \cdots a_n \in F$ s.t.

$a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n = 0 \Rightarrow$ some, of or all the scalars are non-zero

Ex:-1. In vector space $V_3(R)$,
vectors $\alpha_1 = (1,0,0)$, $\alpha_2 = (0,1,0)$, $\alpha_3 = (0,0,1)$
are L.I. as
$a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3 = 0$ where $a_1, a_2, a_3 \in R$
$a_1(1,0,0) + a_2(0,1,0) + a_3(0,0,1) = (0,0,0)$
$\Rightarrow (a_1,0,0) + (0,a_2,0) + (0,0,a_3) = (0,0,0)$
$\Rightarrow (a_1+0+0,\ 0+a_2+0,\ 0+0+a_3) = (0,0,0)$
$\Rightarrow a_1 = 0,\ a_2 = 0,\ a_3 = 0$

Ex:- 2. In $R^3(R)$, vectors $\alpha_1 = (1,3,2)$, $\alpha_2 = (1,-7,-8)$
$\alpha_3 = (2,1,-1)$ are L.D. as
$a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3 = 0$ where $a_1, a_2, a_3 \in R$.
$a_1(1,3,2) + a_2(1,-7,-8) + a_3(2,1,-1) = (0,0,0)$ —(i)
$(a_1 + a_2 + 2a_3,\ 3a_1 - 7a_2 + a_3,\ 2a_1 - 8a_2 - a_3) = (0,0,0)$

$$\Rightarrow \left.\begin{aligned} a_1 + a_2 + 2a_3 &= 0 \\ 3a_1 - 7a_2 + a_3 &= 0 \end{aligned}\right] \Rightarrow \frac{a_1}{1+14} = \frac{a_2}{6-1} = \frac{a_3}{-7-3}$$

$$2a_1 - 8a_2 - a_3 = 0 \qquad \Rightarrow \frac{a_1}{15} = \frac{a_2}{5} = \frac{a_3}{-10}$$

$$\Rightarrow \frac{a_1}{3} = \frac{a_2}{1} = \frac{a_3}{-2}$$

$\Rightarrow$ for $a_1 = 3,\ a_2 = 1,\ a_3 = -2$
eq^n (i) is satisfied.
Hence vectors $\alpha_1, \alpha_2, \alpha_3$ are L.D.

Ex:- A vector space which contains only zero vector is always L.D. as $\forall\ a \in F\ (a \neq 0)$
$a \cdot \alpha = 0$ as $\alpha = 0$

Ex:- A vector space $V_3(R)$ which contains vectors $\alpha_1 = (2,5,6)$, $\alpha_2 = (0,1,2)$, $\alpha_3 = (0,0,4)$ are L.I. as

$$a_1(2,5,6) + a_2(0,1,2) + a_3(0,0,4) = (0,0,0)$$

$$\Rightarrow (2a_1 + 0.a_2 + 0,\ 5a_1 + a_2,\ 6a_1 + 2a_2 + 4a_3) = (0,0,0)$$

$$\Rightarrow 2a_1 = 0,\ 5a_1 + a_2 = 0,\ 6a_1 + 2a_2 + 4a_3 = 0.$$

$$\Rightarrow a_1 = 0,\ a_2 = 0,\ a_3 = 0$$

Theorem:-1. If vector α is the L.C. of vectors $\alpha_1, \alpha_2, \ldots \alpha_n$ then prove that the vectors $\alpha, \alpha_1, \alpha_2, \ldots \alpha_n$ are L.D..

Proof:- given that

$$\alpha = a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n$$ where $a_1, a_2 \ldots a_n \in F$

$$\Rightarrow a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n + (-1).\alpha = 0$$

$\Rightarrow \alpha, \alpha_1, \alpha_2, \ldots, \alpha_n$ are L.D. as the last scalar coffecient of α is non-zero(-1)

H.P.

Th:-2. A set of vectors ~~whi~~ that contains at least one zero vector is L.D.

Proof:- let $T = \{\alpha_1, \alpha_2, \ldots \alpha_n\}$ is the set of vectors s.t. $\alpha_n = 0$ & reminning all are non-zero vectors.

$$0.\alpha_1 + 0.\alpha_2 + \cdots + 0.\alpha_{n-1} + 1.\alpha_n = 0 + 0 + \cdots + 0 + 0 = 0$$

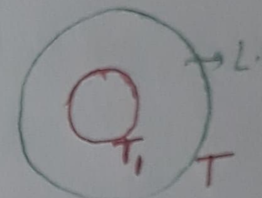
where $0, 1 \in F$

Hence $\sum_{i=1}^{n} a_i \alpha_i = 0 \Rightarrow a_n = 1 \; (\neq 0)$

$a_1 = 0, a_2 = 0 \; \cdots, a_{n-1} = 0$

$\Rightarrow$ T is Linearly Dependent.

Th:-3. Every non empty subset of a L.I. set of vectors is also L.I.



Proof :- let $T = \{\alpha_1, \alpha_2, \cdots, \alpha_n\}$ is the L.I. set of vectors of vector space.

$\Rightarrow \exists$ scalars $a_1, a_2, \cdots, a_n \in F$ s.t.

$a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n = 0 \Rightarrow a_1 = 0, a_2 = 0 \cdots, a_n = 0$ —(i)

Let $T_1 = \{\alpha_1, \alpha_2, \cdots, \alpha_m\}$ be a subset of T, where $1 \le m \le n$

To prove that T_1 is L.I.

$a_1\alpha_1 + a_2\alpha_2 + \cdots + a_m\alpha_m = 0$ where $a_1, a_2, \cdots a_m \in F$

$\Rightarrow \underbrace{a_1\alpha_1 + a_2\alpha_2 + \cdots + a_m\alpha_m}_{} + 0\cdot\alpha_{m+1} + 0\cdot\alpha_{m+2} + \cdots + 0\cdot\alpha_n$

$= 0 + 0 + 0 + \cdots + 0$

$= 0$

$\Rightarrow a_1\alpha_1 + a_2\alpha_2 + \cdots + a_m\alpha_m + 0\cdot\alpha_{m+1} + \cdots + 0\cdot\alpha_n = 0$

$\Rightarrow a_1 = 0, a_2 = 0, \cdots \cdots a_m = 0$ as T is L.I. set. (by (i))

Hence T_1 is L.I. set.

$\Rightarrow$ Every L.I. subset of a L.I. set of vectors is also L.I.