UNIT - III

Elements of Poin' Estimation

statistical Inference

Estimation

Testing of Hypothesis

Point Estimation

Interval Estimation

Consistency

Unbiasedness

Sufficiency

Efficiency

* ~~Statistical Inference~~ :-

~~One of the~~

* Parameter Space :

The set of all possible values of the parameter (say) θ is called the parameter space and it is denoted by Θ i.e. capital θ.

If $X \sim N(\mu, \sigma^2)$ then the parameter space

$$\Theta = \{(\mu, \sigma^2) ; -\infty < \mu < \infty ; \sigma^2 > 0\}$$

* Statistical Inference :- One of the main objective of about a population from the analyses of sample drawn from the population two independent problem of statistical inference are.

1. Estimation

2. Testing of Hypothesis

$X \rightarrow$ Population

$x_1, x_2 - - x_n \rightarrow$ Sample

~~2004~~, ~~2009~~, 2013

Estimation :- The theory of estimation was founded by R.A. Fisher into two groupes.

1. Point Estimation 2. Interval Estimation

~~2004~~, ~~2009~~, 2011, 2012

1. Point Estimation :-

A value of a statistic is used to estimate a given parameter is known as point estimation.

2004, 2009, 2011, 2012, 2013

Estimator :- Any funct^n of the random sample $x_1, x_2 - - - x_n$ that are ~~seen~~ being observed, say $T_n(x_1, x_2 - - - x_n)$ is called a statistic. Clearly, a statistic is a random variable. If it is used to estimate an unknown parameter θ of the dist^n, it is called an estimator.

~~2004~~, ~~2009~~, 2011, 2013

Estimate :- A particular value of the estimator, say $T_n(x_1, x_2 - x_n)$ is called an estimate of θ.

2004, 2008, 2009, 2013

Characteristics of Estimator: (Types of Point Estimation) :-

The following are some of the criteria that should be satisfied by a good estimator.

(i) Unbiasedness (ii) Consistency (iii) Efficiency (iv) Sufficiency

2005, 2006, 2007, 2011, 2012

(1) Unbiasedness :- An estimator $T_n = T(x_1, x_2 - - - x_n)$ is said to be an unbiased estimator of $\gamma(\theta)$ if

If $E(T_n) \neq \gamma(\theta)$ then T_n is said to be biassed estimator of $\gamma(\theta)$

~~2) If $E(T_n) > \gamma(\theta)$ then T_n is~~ said to be +vely biassed

3) If $E(T_n) < \gamma(\theta)$ then T_n is said to be −vely biassed.

by R.A.

$$\boxed{E(T_n) = \gamma(\theta),\ \text{for all } \theta \in \Theta}$$

estimate

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$\ldots x_n$

is called

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of the

$T_n(x_1, x_2, \ldots x_n)$

ation) :-

t should

) Sufficiency

is said

Q1. Prove that $E(\bar{x}_n) = \mu$

Sol^n let $x_1, x_2, \ldots x_n$ be a random sample of size n from a population with mean μ & variance σ^2

i.e. $x \sim N(\mu, \sigma^2)$

Now, Sample mean $= \bar{x}_n = \frac{1}{n}\sum_{i=1}^{n} x_i$

$$E(\bar{x}_n) = \frac{1}{n}\sum_{i=1}^{n} E(x_i)$$

$$= \frac{1}{n}\sum_{i=1}^{n} \mu = \frac{1}{n}\cdot n\cdot\mu \quad [\because E(x_i) = \mu]$$

$$\boxed{E(\bar{x}_n) = \mu}$$

2004, 2006, 2008

Q2. Show that $E(s^2) \neq \sigma^2$ but $E(S'^2) = \sigma^2$

Sol^n we know that

$s^2 =$ Sample mean square $= \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2$

& $S'^2 =$ Population mean square $= \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2$

1) $s^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2$

$= \frac{1}{n}\left[\sum_{i=1}^{n}[(x_i - \mu) - (\bar{x}_i - \mu)]^2\right]$

$= \frac{1}{n}\left[\sum_{i=1}^{n}(x_i - \mu)^2 + \sum_{i=1}^{n}(\bar{x}_i - \mu)^2 - 2\sum(x_i - \mu)(\bar{x} - \mu)\right]$

$= \frac{1}{n}\left[\sum_{i=1}^{n}(x_i - \mu)^2 + n(\bar{x} - \mu)^2 - 2(\bar{x} - \mu)\sum_{i=1}^{n}(x_i - \mu)\right]$

$$s^2 = \frac{1}{n}\left[\sum_{i=1}^{n}(x_i-\mu)^2 + n(\bar{x}-\mu)^2 - 2(\bar{x}-\mu)\cdot n(\bar{x}-\mu)\right]$$

$\because \sum(x_i-\mu) = \sum x_i - \sum\mu = n\bar{x} - n\mu = n(\bar{x}-\mu)$

$$= \frac{1}{n}\left[\sum_{i=1}^{n}(x_i-\mu)^2 + n(\bar{x}-\mu)^2 - 2n(\bar{x}-\mu)^2\right]$$

$$s^2 = \frac{1}{n}\left[\sum_{i=1}^{n}(x_i-\mu)^2 - n(\bar{x}-\mu)^2\right]$$

taking expectation both side.

$$E(s^2) = E\left[\frac{1}{n}\left\{\sum_{i=1}^{n}(x_i-\mu)^2 - n(\bar{x}-\mu)^2\right\}\right]$$

$$= \frac{1}{n}\left[\sum_{i=1}^{n}E(x_i-\mu)^2 - nE(\bar{x}-\mu)^2\right]$$

$$= \frac{1}{n}\left[\sum_{i=1}^{n}E[x_i-E(x_i)]^2 - nV(\bar{x})\right]$$

$\because \mu = E(x_i)$

& $V(x) = E[x-E(x)]^2$

$V(\bar{x}) = E[\bar{x}-\mu]^2$

& $V(\bar{x}) = \frac{\sigma^2}{n}$

$$= \frac{1}{n}\left[\sum_{i=1}^{n}V(x_i) - nV(\bar{x})\right]$$

$$= \frac{1}{n}\sum_{i=1}^{n}\sigma^2 - V(\bar{x})$$

$$= \frac{1}{n}\cdot\sigma^2\sum_{i=1}^{n}(1) - \frac{\sigma^2}{n} = \frac{1}{n}\cdot\sigma^2\cdot n - \frac{\sigma^2}{n}$$

$$E(s^2) = \sigma^2\left(1-\frac{1}{n}\right) \neq \sigma^2$$

i.e. $\boxed{E(s^2) \neq \sigma^2}$ (H.P.)

& $s'^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i-\bar{x})^2$

$$= \frac{1}{n-1}\sum_{i=1}^{n}\left[(x_i-\mu)-(\bar{x}-\mu)\right]^2$$

$$= \frac{1}{n-1}\left[\sum_{i=1}^{n}(x_i-\mu)^2 + \sum_{i=1}^{n}(\bar{x}-\mu)^2 - 2\sum_{i=1}^{n}(x_i-\mu)(\bar{x}-\mu)\right]$$

$$s'^2 = \frac{1}{n-1}\left[\sum_{i=1}^{n}(x_i-\mu)^2 + n(\bar{x}-\mu)^2 - 2(\bar{x}-\mu)\cdot\sum_{i=1}^{n}(x_i-\mu)\right]$$

$\because \Sigma(x_i-\mu) = \Sigma x_i - \Sigma\mu = n\bar{x} - n\mu = n(\bar{x}-\mu)$

$$= \frac{1}{n-1}\left[\sum_{i=1}^{n}(x_i-\mu)^2 + n(\bar{x}-\mu)^2 - 2(\bar{x}-\mu)\cdot n(\bar{x}-\mu)\right]$$

$$s'^2 = \frac{1}{n-1}\left[\sum_{i=1}^{n}(x_i-\mu)^2 - n(\bar{x}-\mu)^2\right]$$

taking expectation both sides

$$E(s'^2) = E\left[\frac{1}{n-1}\left\{\sum_{i=1}^{n}(x_i-\mu)^2 - n(\bar{x}-\mu)^2\right\}\right]$$

$$= \frac{1}{n-1}\left[\sum_{i=1}^{n}E(x_i-\mu)^2 - nE(\bar{x}-\mu)^2\right]$$

$$= \frac{1}{n-1}\left[\sum_{i=1}^{n}E\{x_i-E(x_i)\}^2 - nV(\bar{x})\right]$$

$\because$ 1) $\mu = E(x_i)$
2) $V(x) = E[x-E(x)]^2$
$V(\bar{x}) = E(\bar{x}-\mu)^2$
$V(\bar{x}) = \frac{\sigma^2}{n}$

$$= \frac{1}{n-1}\left[\sum_{i=1}^{n}V(x_i) - nV(\bar{x})\right]$$

$$= \frac{1}{n-1}\left[\sum_{i=1}^{n}\sigma^2 - \cancel{n}\cdot\frac{\sigma^2}{\cancel{n}}\right]$$

$$= \frac{1}{n-1}\left[\sigma^2\cdot\sum_{i=1}^{n}(1) - \sigma^2\right] = \frac{1}{n-1}\left[\sigma^2 n - \sigma^2\right]$$

$$= \frac{\sigma^2}{\cancel{n-1}}\cancel{(n-1)}$$

$$\boxed{E(s'^2) = \sigma^2}$$

H.P.

Q.3. $x_1, x_2, \ldots x_n$ is a random sample ~~from~~ from a normal population $N(\mu, 1)$. Show that $t = \frac{1}{n}\sum_{i=1}^{n}x_i^2$ is an unbiased estimator of μ^2+1 i.e. $\boxed{E(t) = \mu^2+1}$

Sol^n $\because$ $x_i \sim N(\mu, 1)$

i.e. $E(x_i) = \mu$ & $V(x_i) = 1$

$E(x_i^2) = V(x_i) + [E(x_i)]^2$ $\quad [\because V(x) = E(x^2) - [E(x)]^2]$

$E(x_i^2) = 1 + \mu^2 \rightarrow (1)$

Now $t = \frac{1}{n}\sum_{i=1}^{n} x_i^2$ taking expectation both sides

$$E(t) = E\left[\frac{1}{n}\sum_{i=1}^{n} x_i^2\right]$$

$$= \frac{1}{n}\sum_{i=1}^{n} E(x_i^2)$$

$$= \frac{1}{n}\sum_{i=1}^{n} (\mu^2+1) \quad \text{(from eq.}^n \text{(1))}$$

$$= \frac{1}{n}\cdot(\mu^2+1)\cdot\sum_{i=1}^{n}(1) = \frac{1}{\cancel{n}}(\mu^2+1)\cdot\cancel{n}$$

$$\boxed{E(t) = \mu^2+1}$$

(I.P.)

Hence t is an unbiased estimator of $1+\mu^2$.

Q.4. Show that $\left[\frac{\Sigma x_i(\Sigma x_i - 1)}{n(n-1)}\right]$ is an unbiased estimate of θ^2 for the sample $x_1, x_2, \ldots x_n$ drawn on x which takes the values 1 and 0 with respective probabilities θ and $1-\theta$.

Sol^n Since $x_1, x_2, \ldots x_n$ be a random sample from Bernoulli population with parameter $\theta \in (1-\theta)$

Hence $T = \sum_{i=1}^{n} x_i \sim B(n, \theta) \Rightarrow E(T) = n\theta$.

& $V(T) = n\theta(1-\theta)$

for Bernoulli popul^n $= B(n,p)$
$E(X) = np$
$V(X) = npq$
$p = \theta$ & $q = 1-\theta$

$\therefore$ According to que.

$$E\left[\frac{\Sigma x_i(\Sigma x_i - 1)}{n(n-1)}\right] = E\left[\frac{T(T-1)}{n(n-1)}\right] = E\left[\frac{T^2-T}{n(n-1)}\right]$$

$$= \frac{1}{n(n-1)}\left[E\{T^2-T\}\right] = \frac{1}{n(n-1)}\left[E(T^2) - E(T)\right]$$

$\because V(T) = E(T^2) - [E(T)]^2$

$\Rightarrow E(T^2) = V(T) + [E(T)]^2$

$= n\theta(1-\theta) + (n\theta)^2$

$E(T^2) = n\theta(1-\theta) + n^2\theta^2$

$$= \frac{1}{n(n-1)}\left[n\theta(1-\theta) + n^2\theta^2 - n\theta\right]$$

$$= \frac{1}{n(n-1)}\left[\cancel{n\theta} - n\theta^2 + n^2\theta^2 - \cancel{n\theta}\right]$$

$$= \frac{1}{n(n-1)}\left[n^2\theta^2 - n\theta^2\right] = \frac{1}{\cancel{n(n-1)}} \cdot \cancel{n}\theta^2\cancel{(n-1)}$$

$$E\left[\frac{\Sigma x_i(\Sigma x_i - 1)}{n(n-1)}\right] = \theta^2$$

Hence, $\frac{\Sigma x_i(\Sigma x_i - 1)}{n(n-1)}$ is an unbiased estimator of θ^2.

2005, 2006, 2007, 2011, 2012

(ii) Consistency :-

An estimator $T_n = T(x_1, x_2, \ldots, x_n)$ based on random sample of size n is said to be consistent estimator of $\gamma(\theta)$, $\theta \in \Theta$ (Parameter Space). If T_n converges to $\gamma(\theta)$ in probability i.e. $T_n \xrightarrow{P} \gamma(\theta)$ as $n \to \infty$.

Or

In other words, T_n is a consistent estimator of $\gamma(\theta)$ if for every $\epsilon > 0$, $\eta > 0$, there exists a positive integer $n \geq m(\epsilon, \eta)$ such that $P[|T_n - \gamma(\theta)| < \epsilon] \to 1$ as $n \to \infty$

$$\Rightarrow P[|T_n - \gamma(\theta)| < \epsilon] > 1 - \eta \;;\; \forall\, n \geq m$$

where m is some very large value of n.

$= E(x^2) - [E(x)]^2$

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$1-\theta)$

for Bernoulli popul^n $= B(n,p)$

$E(x) = np$

$V(x) = npq$

$p = \theta$ & $q = 1-\theta$

Properties of Consistency -

1. Invariance Property of Consistent Estimators :-

Theorm :-

If T_n is a consistent estimator of $\gamma(\theta)$ and $\psi\{\gamma(\theta)\}$ is a continious function of $\gamma(\theta)$, then $\psi(T_n)$ is a consistent estimator of $\psi\{\gamma(\theta)\}$.

Proof :-

Since, T_n is a consistent estimator of $\gamma(\theta)$

i.e. $T_n \xrightarrow{P} \gamma(\theta)$ as $n \to \infty$

i.e. for every $\varepsilon > 0$, $\eta > 0$ $\exists$ a possitive integer $n \geq m(\varepsilon, \eta)$ such that

$$P[|T_n - \gamma(\theta)| < \varepsilon] > 1 - \eta \quad \forall\ n \geq m \longrightarrow ①$$

Since, $\psi(\cdot)$ is a continuous function for every $\varepsilon > 0$, $\exists$ a positive ε_1 such that $|\psi(T_n) - \psi\{\gamma(\theta)\}| < \varepsilon_1$ whenever $|T_n - \gamma(\theta)| < \varepsilon$

i.e. $\underbrace{|T_n - \gamma(\theta)| < \varepsilon}_{A} \Rightarrow \underbrace{|\psi(T_n) - \psi\{\gamma(\theta)\}| < \varepsilon_1}_{B} \to ②$

for two events A and B

if $A \Rightarrow B$, then $A \subseteq B \Rightarrow P(A) \leq P(B)$

$$\Rightarrow P(B) \geq P(A) \to ③$$

by eq^n ② & ③, we get -

$$P(B) \geq P(A)$$

$$P[|\psi(T_n) - \psi\{\gamma(\theta)\}| < \varepsilon_1] \geq P[|T_n - \gamma(\theta)| < \varepsilon]$$

$$P[|\psi(T_n) - \psi\{\gamma(\theta)\}| < \varepsilon_1] \geq 1 - \eta, \quad n \geq m \quad (\text{from eq}^n ①)$$

i.e. $\psi(T_n) \xrightarrow{P} \psi\{\gamma(\theta)\}$; as $n \to \infty$

or $\psi(T_n)$ is a consistent estimator of $\gamma(\theta)$.

2005, 2007, 2008

~~Q.~~ Sufficient Condition for Consistency :-

Theorem :-

Let T_n be a sequence of estimators such that for all $\theta \in$

(i) $E_\theta(T_n) \longrightarrow \gamma(\theta)$, $n \to \infty$

(ii) $Var_\theta(T_n) \longrightarrow 0$, as $n \to \infty$

Then T_n is a consistent estimator of $\gamma(\theta)$.

Proof :-

we have to prove that T_n is a consistent estimator of $\gamma(\theta)$

i.e. $T_n \xrightarrow{P} \gamma(\theta)$; as $n \to \infty$

$$P[|T_n - \gamma(\theta)| < \varepsilon] > 1 - \eta \;;\; \forall\, n \geq m(\varepsilon, \eta)$$

where ε and η are arbitrarily small positive numbers and m is some large values of n.

Applying Chebychev's inequality to the statistic T_n, we get

$$P[|T_n - E_\theta(T_n)| \leq \delta] \geq 1 - \frac{Var_\theta(T_n)}{\delta^2} \longrightarrow (1)$$

we have

$$|T_n - \gamma(\theta)| = |T_n - E_\theta(T_n) + E_\theta(T_n) - \gamma(\theta)| \leq |T_n - E_\theta(T_n)| + |E_\theta(T_n) - \gamma(\theta)|$$

Now,

$$\underbrace{|T_n - E_\theta(T_n)| \leq \delta|}_{A} \Rightarrow \underbrace{|T_n - \gamma(\theta)| \leq \delta + |E_\theta(T_n) - \gamma(\theta)|}_{B}$$

we known that if $A \Rightarrow B$ then $P(A) \leq P(B)$

i.e. $P(B) \geq P(A)$

$$P[|T_n - \gamma(\theta)| \le \delta + |E_\theta(T_n) - \gamma(\theta)|] \geq P[|T_n - E_\theta(T_n)| \le \delta]$$

$$\geq 1 - \frac{Var_\theta(T_n)}{\delta^2} \quad \text{(from eq}^n\text{ ①)} \rightarrow ②$$

We are given $E_\theta(T_n) \longrightarrow \gamma(\theta) \;\; \forall \theta \in \Theta$ as $n \to \infty$

$$|E_\theta(T_n) - \gamma(\theta)| \le \delta_1, \quad \forall\, n \geq n_0(\delta_1) \rightarrow ③$$

Also given that

$$Var_\theta(T_n) \to 0 \text{ as } n \to \infty$$

$$\therefore \boxed{\frac{Var_\theta(T_n)}{\delta^2} \le \eta} \quad \forall\, n \geq n_0'(\eta) \rightarrow ④$$

where η is arbitrarily small positive number.

putting the value by eq^n ③ & ④ in eq^n ② we get

$$P[|T_n - \gamma(\theta)| \le \delta + \delta_1] \geq 1 - \eta \;;\; n \geq m(\delta, \delta_1, \eta)$$

$$\Rightarrow P[|T_n - \gamma(\theta)| \le \varepsilon] \geq 1 - \eta \;;\; n \geq m, \text{ where } \varepsilon = \delta + \delta_1$$

i.e. by def^n of consistency we have

$$\Rightarrow \boxed{T_n \xrightarrow{P} \gamma(\theta)} \text{ as } n \to \infty$$

$\therefore$ T_n is a consistent estimator of $\gamma(\theta)$.

2011, 2012

(iii) Efficiency :-

2003, 2004 **Efficient Estimators :-**

Let T_1 and T_2 are two consistent estimators of a certain parameter θ, we have -

$$V(T_1) < V(T_2) \;;\; \text{for all } n$$

then T_1 is

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then T_1 is more efficient than T_2 for all sample sizes.

for example - $Var.(\bar{x}) = \frac{\sigma^2}{n}$ $[\bar{x} \sim N(\mu, \sigma^2/n)]$

$Var(mid) = \frac{\pi}{2} \cdot \frac{\sigma^2}{n} = 1.52 \frac{\sigma^2}{n}$

$V(\bar{x}) < V(mid)$

Hence, sample mean is more efficient then median

In others words :-

$$\boxed{\text{Efficiency} \propto \frac{1}{\text{Variance}}}$$

Most Efficient Estimators :-

If in a class of consistent estimators for a parameter, there, exists one whose sampling variance is less then that of any such estimator. it is called the most efficient estimator. whenever such an estimator exists, it provided a criterion for measurment of efficiency of the other estimators.

2004

definition of Efficiency :-

If T_1 is the most efficient estimator with variance V_1 and T_2 is any other estimator with variance V_2, then the efficiency E of T_2 is defined as.

mee

$$\boxed{E = \frac{V_1}{V_2}}$$

V_1 → known

V_2 → unknown

obviously, E cannot exceed unity

Q.5. A random sample $(X_1, X_2, X_3, X_4, X_5)$ of size $n=5$ is drawn from a normal population with unknown mean μ. Consider the following estimators to estimate μ.

(i) $t_1 = \frac{X_1+X_2+X_3+X_4+X_5}{5}$ (ii) $t_2 = \frac{X_1+X_2}{2}+X_3$, (iii) $t_3 = \frac{2X_1+X_2+\lambda X_3}{3}$

where λ is such that t_3 is an unbiased estimator of μ.

Find λ. Also check whether t_1 & t_2 are unbiased and check which is most efficient estimator among t_1, t_2 & t_3.

Sol^n. We are given that

$E(X_i) = \mu$, $Var(X_i) = \sigma^2$ (say) $Cov(X_i, X_j) = 0$ $(i \neq j)$ $\longrightarrow (1)$

(i) Check whether t_1 & t_2 are unbiased

1) $t_1 = \frac{X_1+X_2+X_3+X_4+X_5}{5}$

Taking expectation both sides

$E(t_1) = E\left[\frac{X_1+X_2+X_3+X_4+X_5}{5}\right]$

$= \frac{1}{5}\left[E(X_1)+E(X_2)+E(X_3)+E(X_4)+E(X_5)\right]$

$E(t_1) = \frac{1}{5} \cdot 5\mu = \mu$ (by eq^n (1))

$\Rightarrow t_1$ is an unbiased estimator of μ.

2) $t_2 = \frac{X_1+X_2}{2} + X_3$

: 5 is drawn
ean μ. Consider

$= \frac{2x_1+x_2+\lambda x_3}{3}$

estimator of μ.

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ator among t_1, t_2 & t_3.

$j) = 0 \quad (i \neq j)$
$\longrightarrow$ (1)

$E(x_5)]$
n (1)

taking expectation both sides

$$E(t_2) = E\left(\frac{x_1+x_2}{2}\right) + E(x_3)$$

$$= \frac{1}{2}[E(x_1) + E(x_2)] + E(x_3)$$

$$= \frac{1}{\cancel{2}} \cdot \cancel{2}\mu + \mu \qquad \text{(by (1))}$$

$$E(t_2) = 2\mu \neq \mu$$

$\Rightarrow$ t_2 is not an unbiased estimator of μ.

(ii) Find λ –

$$t_3 = \frac{2x_1+x_2+\lambda x_3}{3}$$

According to que. t_3 is an unbiased estimator of
i.e. $E(t_3) = \mu$

$$E(t_3) = E\left[\frac{2x_1+x_2+\lambda x_3}{3}\right]$$

$$\mu = \frac{1}{3}[E(2x_1) + E(x_2) + E(\lambda x_3)]$$

$$= \frac{1}{3}[2E(x_1) + E(x_2) + \lambda . E(x_3)]$$

$$= \frac{1}{3}[2\mu + \mu + \lambda\mu] \qquad \text{(by (1))}$$

$$\mu = \frac{1}{3}(3\mu + \lambda\mu)$$

$$3\mu = 3\mu + \lambda\mu$$

$$\Rightarrow \lambda\mu = 0$$

$$\Rightarrow \boxed{\lambda = 0}$$

$$\Rightarrow \boxed{t_3 = \frac{2x_1+x_2}{3}}$$

Check most efficient estimator :-

1) $t_1 = \frac{x_1+x_2+x_3+x_4+x_5}{5}$

$$V(t_1) = V\left[\frac{x_1+x_2+x_3+x_4+x_5}{5}\right]$$

$$= \frac{1}{25}\left[V(x_1)+V(x_2)+V(x_3)+V(x_4)+V(x_5)\right] \quad [\because V(ax) = a^2V(x)]$$

$$= \frac{1}{25}\left[\sigma^2+\sigma^2+\sigma^2+\sigma^2+\sigma^2\right] \quad (\text{by eq.}^n (1))$$

$$= \frac{1}{\cancel{25}_5}\cdot\cancel{5}\sigma^2 = \frac{\sigma^2}{5} = 0.2\sigma^2$$

2) $t_2 = \frac{x_1+x_2}{2} + x_3$

$$V(t_2) = V\left[\frac{x_1+x_2}{2} + x_3\right]$$

$$= \frac{1}{4}\left[V(x_1)+V(x_2)\right] + V(x_3)$$

$$= \frac{1}{4}\left[\sigma^2+\sigma^2\right] + \sigma^2$$

$$V(t_2) = \frac{\sigma^2}{2} + \sigma^2 = \frac{3\sigma^2}{2} = 1.5\sigma^2$$

3) $v(t_3) = \frac{2x_1+x_2}{3}$

$$V(t_3) = V\left[\frac{2x_1+x_2}{3}\right]$$

$$= \frac{1}{9}\left[4V(x_1)+V(x_2)\right]$$

$$= \frac{1}{9}\left[4\sigma^2+\sigma^2\right]$$

$$= \frac{5}{9}\sigma^2 = 0.5\sigma^2$$

Since $V(t_1)$ is minimum ; so t_1 is the most efficient estimator

2004, 2005
(iv) Sufficien
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2004, 2005
Ex. → Let
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2004, 2005, 2006, 2007, 2008, 2009, 2011

(iv) Sufficiency :-

An estimator is said to be sufficient for a parameter, if it contains all the information in the sample regarding the parameter.

If $T = t(x_1, x_2, \ldots x_n)$ is an estimator of a parameter θ, based on a sample $x_1, x_2, \ldots x_n$ of size n from the population with density $f(x, \theta)$ such that the conditional distribution of $x_1, x_2, \ldots x_n$ given T, is independent of θ, then T is sufficient estimator for θ.

2004, 2005

Ex. → Let $x_1, x_2, \ldots x_n$ be a random sample from a Bernoulli population with parameter 'p', $0 < p < 1$ i.e.

$$x_i = \begin{cases} 1, & \text{with prob. } p \\ 0, & \text{with prob. } q = 1-p \end{cases}$$

Then $T = t(x_1, x_2, \ldots x_n) = x_1 + x_2 + x_3 + \ldots + x_n \sim B(n, p)$

$\therefore P(T = k) = {}^nC_k \, p^k (1-p)^{n-k} \quad ; k = 0, 1, 2, \ldots n$

The conditional distⁿ of $(x_1, x_2, \ldots x_n)$ given T is

$$P(x_1 \cap x_2 \cap x_3 \cap \ldots \cap x_n | T = k) = \frac{P(x_1 \cap x_2 \cap x_3 \cap \ldots x_n \cap T = k)}{P(T = k)}$$

[by conditional prob. $P(A|B) = \dfrac{P(A \cap B)}{P(B)}$]

$$= \begin{cases} \dfrac{p^k (1-p)^{n-k}}{{}^nC_k \, p^k (1-p)^{n-k}} = \dfrac{1}{{}^nC_k} \\ 0 & \text{if } \sum x_i \neq k \end{cases}$$

Since, this does not depend on p, $T = \sum_{i=1}^{n} x_i$ is sufficient for 'p'.

2004, 2005, 2008, 2009, 2011, 2012

Imp.

* Factorization Theorem (Neymann) :-

The necessary and sufficient condition for a distⁿ to admit sufficient statistic is provided by the "factorization theorem" due to Neymann.

Statement :-

If $T = t(x)$ is sufficient for θ if and only if the joint density function L (say) of the sample values can be expressed in the form

jdf

$$\boxed{L = g_\theta[t(x)] \cdot h(x)}$$

where $g_\theta[t(x)]$ depends on θ and x only through the value of $t(x)$ and $h(x)$ is independent of θ.

Q.6. Let $x_1, x_2, \ldots x_n$ be a random sample from a population with p.d.f.

$$f(x, \theta) = \theta x^{\theta - 1} \; ; \; 0 < x < 1 \; , \; \theta > 0$$

Show that $t = \prod_{i=1}^{n} x_i$ is sufficient for θ.

Solⁿ By Factorization theorem is given by-

$$L = g_\theta[t(x)] \cdot h(x)$$

$$= \prod_{i=1}^{n} f(x_i, \theta)$$

$$= \prod_{i=1}^{n} (\theta x_i^{\theta-1})$$

$$L = \theta^n \left(\prod_{i=1}^{n} x_i^{\theta-1}\right)$$

$$= \underbrace{\theta^n \cdot \left(\prod_{i=1}^{n} x_i^{\theta}\right)}_{\uparrow} \cdot \left(\frac{1}{\prod_{i=1}^{n} x_i}\right)$$

$$L = g_\theta(t, \theta) \cdot h(x)$$

Hence by factorization theorem

$$t_1 = \prod_{i=1}^{n} x_i \text{ is sufficient estimator of } \theta.$$

2007, 2009, 2012

Q7. Let $x_1, x_2, \ldots x_n$ be a random sample from $N(\mu, \sigma^2)$ popu

Find sufficient estimators for μ and σ^2.

Sol.^{n} According to quest^n $\Rightarrow$ $\theta = (\mu, \sigma^2)$; $-\infty < \mu < \infty$, $0 \leq \sigma^2 <$

Then Joint dist^n function $L = \prod_{i=1} f(x_i, \theta)$

$$L = \prod_{i=1}^{n} \left[\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x_i-\mu}{\sigma}\right)^2}\right] \quad \text{[by def}^n \text{ of normal dis}$$

$$= \left(\frac{1}{\sigma\sqrt{2\pi}}\right)^n \prod_{i=1}^{n} \left[e^{-\frac{1}{2}\left(\frac{x_i-\mu}{\sigma}\right)^2}\right]$$

$$= \left(\frac{1}{\sigma\sqrt{2\pi}}\right)^n e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2} \quad \left[\because \prod_{i=1}^{n} e^{-x_i} = e^{-\sum x_i}\right]$$

$$= \left(\frac{1}{\sigma\sqrt{2\pi}}\right)^n e^{-\frac{1}{2\sigma^2}\left[\sum_{i=1}^{n} x_i^2 - 2\mu\sum_{i=1}^{n} x_i + n\mu^2\right]}$$

$$L = g_\theta[t(x)] \cdot h(x)$$

where $g_\theta[t(x)] = \left(\frac{1}{\sigma\sqrt{2\pi}}\right)^n e^{-\frac{1}{2\sigma^2}\left[\sum_{i=1} x_i^2 - 2\mu\sum_{i=1} x_i + n\mu^2\right]}$ & $h(x) = 1$

here joint sufficiency. $t(x) = [t_1(x), t_2(x)]$

$t(x) = [\Sigma x_i, \Sigma x_i^2]$

here $g_\theta[t(x)] = \left(\frac{1}{\sigma\sqrt{2\pi}}\right)^n e^{-\frac{1}{2\sigma^2}[t_2(x) - 2\mu t_1(x) + n\mu^2]}$

here, $t_1(x) = \Sigma x_i$ is sufficient estimator for μ;

and $t_2(x) = \Sigma x_i^2$ is sufficient estimator for σ^2.

2012, 2004, 2006, 2007, 2009, 2013 UNIT - IV

*** Bias :-**

An estimator that is not unbiased is called biased estimator.

$$E(T_n) \neq \gamma(\theta)$$

then $\boxed{bias = E(T_n) - \gamma(\theta)}$

1) Bias will be positive if :-

$$\left.\begin{array}{l} E(T_n) - \gamma(\theta) > 0 \\ \Rightarrow E(T_n) > \gamma(\theta) \end{array}\right\} \text{Positive Bias}$$

Most

2) Bias will be negative if :-

$$\left.\begin{array}{l} E(T_n) - \gamma(\theta) < 0 \\ \Rightarrow E(T_n) < \gamma(\theta) \end{array}\right\} \text{Negative Bias}$$

2012, 2006, 2007, 2009, 2011, 2013

*** Mean Square Error (M.S.E.) :-**

The mean square error of an estimator T_n is given by :-

$$M.S.E. = E[T_n - \gamma(\theta)]^2$$

where, T_n is ~~so~~ not an unbiased estimator.

If there are more then one unbiased estimators, the problem arises which one to choose out of the class of unbiased estimators.

Also we required the sampling variance as well as bias should be minimum. To short out the problem, we take the help of M.S.E.

2004, 2006, 2007, 2009, 2013

* Variance of an estimator :-

If T_n is an estimator of $\gamma(\theta)$

then, variance is given by -

$$Var.(T_n) = E[T_n - E(T_n)]^2$$

$$Var.(T_n) = E[T_n - \gamma(\theta)]^2 \quad \forall\ \theta \in \Theta \qquad [\because E(T_n) = \gamma(\theta) \text{ when } T_n \text{ is unbiased estimator}]$$

2004, 2005, 2006, 2007, 2008, 2009, 2012, 2013

* Relation Between Bias, Mean Square Error (M.S.E.) and Variance of an estimator :-

$\because$ we know that

$$M.S.E. = E[T_n - \gamma(\theta)]^2$$

$$= E[T_n - E_\theta(T_n) + E_\theta(T_n) - \gamma(\theta)]^2$$

$$= E[\{T_n - E_\theta(T_n)\}^2 + \{E_\theta(T_n) - \gamma(\theta)\}^2 + 2\{T_n - E_\theta(T_n)\}\{E_\theta(T_n) - \gamma(\theta)\}]$$

$$= E\{T_n - E_\theta(T_n)\}^2 + E\{E_\theta(T_n) - \gamma(\theta)\}^2 + 2E[\{T_n - E_\theta(T_n)\}\{E_\theta(T_n) - \gamma(\theta)\}]$$

$$= \underline{E\{T_n - E_\theta(T_n)\}^2} + E\{E_\theta(T_n) - \gamma(\theta)\}^2 + 2E[\{T_n - E_\theta(T_n)\}\{E_\theta(T_n) - E_\theta(T_n)\}]$$

$$[\because E_\theta(T_n) = \gamma(\theta)]$$

$$= E\{T_n - E_\theta(T_n)\}^2 + E\{E_\theta(T_n) - \gamma(\theta)\}^2$$

$$\boxed{M.S.E. = Var(T_n) + (Bias)^2}$$

T_n is an unbiased estimator of $\gamma(\theta)$

i.e. $E(T_n) = \gamma(\theta)$

and $Var.(T_n)$ is [illegible]

2004, 2006, 2007, 2008, 2009, 2011, 2012

* Minimum Variance Unbiased Estimator (M.V.U.E.) :-

If a statistic $T = T(x_1, x_2, \ldots\ldots, x_n)$ based on sample of size n is such that:

1) T is unbiased for $\gamma(\theta)$; $\forall\ \theta \in \Theta$ and

2) If T has the smallest variance among the class of all unbiased estimators of $\gamma(\theta)$, then T is called the minimum variance unbiased estimator (M.V.U.E.) of $\gamma(\theta)$.

More precisely.

T is M.V.U.E. of $\gamma(\theta)$ if.

1) $E_\theta(T) = \gamma(\theta) \quad \forall\ \theta \in \Theta$

and 2) $Var_\theta(T) \leq Var_\theta(T') \quad \forall\ \theta \in \Theta$

where T' is any other unbiased estimator of $\gamma(\theta)$.

2006, ~~20~~, 2013

* Theorems or Properties of M.V.U.E. :-

2004, 2008, 2011 Theorem 1. - An M.V.U.E. is unique in the sense that if T_1 & T_2 are M.V.U.E for $\gamma(\theta)$, then $T_1 = T_2$, almost sure.

Proof :- ∵ we have given that T_1 & T_2 be a unbiased estimator of $\gamma(\theta)$

$$E_\theta(T_1) = E_\theta(T_2) = \gamma(\theta) \longrightarrow (1) \quad \forall\ \theta \in \Theta$$

and $Var_\theta(T_1) = Var_\theta(T_2) \quad \forall\ \theta \in \Theta \longrightarrow (2)$

Now, we consider a new estimator

$$T = \frac{T_1 + T_2}{2}, \longrightarrow (3)$$ which is also unbiased,

Since

$$E(T) = \frac{E(T_1) + E(T_2)}{2}$$

.E.) :-

$\ldots, x_n)$

and

the class of

is called the

M.V.U.E.) of $\gamma(\theta)$.

tor of $\gamma(\theta)$.

e sense that

T_2, almost sure

biased estimator of $\gamma(\theta)$

$\theta \in \Theta$

$\rightarrow (2)$

unbiased,

$V(ax) = a^2 V(x)$

$E(T) = \frac{1}{2}(\gamma(\theta) + \gamma(\theta)) = \frac{1}{\cancel{2}} \cdot \cancel{2}\gamma(\theta)$ (by (1))

$\boxed{E(T) = \gamma(\theta)} \rightarrow (4)$

taking Variance of eq^n (3) on both sides

$V(T) = \cancel{V(T_1)+} V\left[\frac{1}{2}(T_1+T_2)\right]$

$= \frac{1}{4} V(T_1+T_2)$

$= \frac{1}{4}\left[V(T_1) + V(T_2) + 2\,\mathrm{Cov}(T_1, T_2)\right]$

$V(T) = \frac{1}{4}\left[V(T_1) + V(T_2) + 2\rho\sqrt{V(T_1)\cdot V(T_2)}\right]$ $\left[\because \rho = \frac{\mathrm{Cov}(x,y)}{\sqrt{V(x)\cdot V(y)}} \Rightarrow \mathrm{Cov}(x,y) = \rho\sqrt{V(x)\cdot V(y)}\right]$

where ρ is Karl Pearson's Coefficient of correlation between T_1 & T_2.

$V(T) = \frac{1}{4}\left[V(T_1) + V(T_1) + 2\rho\sqrt{V(T_1)\cdot V(T_1)}\right]$ (by eq^n (2))

$= \frac{1}{4}\left[2V(T_1) + 2\rho V(T_1)\right]$

$= \frac{1}{\cancel{4}_2} \cancel{2} V(T_1)(1+\rho)$

$\boxed{V(T) = \frac{1}{2} V(T_1)\cdot(1+\rho)} \rightarrow (5)$

Since, T_1 is the M.V.U.E.

hence $V(T) \geq V(T_1)$, T is any other estimator

$\frac{1}{2} V(T_1)\cdot(1+\rho) \geq V(T_1)$ (by (5))

$\frac{1}{2}(1+\rho) \geq 1$

$\Rightarrow 1+\rho \geq 2$

$\boxed{\Rightarrow \rho \geq 1}$

Since we must have $|\rho| \leq 1$ hence take $\rho = 1$. if T_1 & T_2 must have a linear relation between them

$$T_1 = \alpha + \beta T_2 \longrightarrow (6)$$

where, α and β are constants independent of $x_1, x_2, \ldots, x_n$ but may depended on θ. i.e. $\alpha = \alpha(\theta)$ and $\beta = \beta(\theta)$ $\longrightarrow (7)$

taking Expectation of eq.^{n} (6)

$$E(T_1) = E(\alpha) + E(\beta T_2)$$

$$\gamma(\theta) = \alpha + \beta E(T_2) = \alpha + \beta \gamma(\theta) \quad (\text{by eq}^n (1))$$

$$\theta = \alpha + \beta \theta \longrightarrow (8) \quad (\text{by eq}^n (7))$$

taking Variance of eq^n (6)

$$V(T_1) = V(\alpha) + V(\beta T_2)$$

$$V(T_1) = \beta^2 V(T_2)$$

$$\cancel{V(T_1)} = \beta^2 \cancel{V(T_1)} \quad (\text{by eq}^n (1))$$

$$\beta^2 = 1$$

$$\Rightarrow \boxed{\beta = \pm 1}$$

But since, $\rho(T_1, T_2) = +1$, the Coefficient of regression of T_1 on T_2 must be positive

$$\therefore \boxed{\beta = 1} \Rightarrow \boxed{\alpha = 0} \quad (\text{by eq}^n (8))$$

Substituting this values in eq^n (6), we get

$$\boxed{T_1 = T_2} \quad (H.P)$$

Theorem 2:- Let T_1 & T_2 be unbiased estimators of $\gamma(\theta)$ with efficiencies e_1 & e_2 respectively and $\rho = \rho_0$ be the correlation Coefficient between them. Then show that

$$\sqrt{e_1 e_2} - \sqrt{(1-e_1)(1-e_2)} \leq \rho \leq \sqrt{e_1 e_2} + \sqrt{(1-e_1)(1-e_2)}$$

Proof :- Let T be a M.V.U.E. of $\gamma(\theta)$ and we are given.

$$E(T_1) = E(T_2) = \gamma(\theta) \quad \forall \ \theta \in \Theta$$

[given that T_1 & T_2 be unbiased estimators]

and $e_1 = \dfrac{V(T)}{V(T_1)} = \dfrac{V}{V_1} \Rightarrow V_1 = \dfrac{V}{e_1} \longrightarrow (1)$

similarly $e_2 = \dfrac{V(T)}{V(T_2)} = \dfrac{V}{V_2} \Rightarrow V_2 = \dfrac{V}{e_2} \longrightarrow (2)$

Let us consider another estimator which is also unbiased.

$$E(T_3) = \gamma(\theta) \Rightarrow T_3 = \lambda T_1 + \mu T_2 \longrightarrow (3)$$

taking expectation on both sides

$$E(T_3) = \lambda E(T_1) + \mu E(T_2)$$
$$= \lambda \gamma(\theta) + \mu \gamma(\theta)$$
$$\gamma(\theta) = (\lambda + \mu) \gamma(\theta)$$
$$\boxed{\lambda + \mu = 1} \longrightarrow (4)$$

taking variance of eqn (3) on both sides

$$V(T_3) = V[\lambda T_1 + \mu T_2]$$
$$= \lambda^2 V(T_1) + \mu^2 V(T_2) + 2\lambda\mu \, Cov(T_1, T_2)$$
$$= \lambda^2 V(T_1) + \mu^2 V(T_2) + 2\lambda\mu \cdot \rho \cdot \sqrt{V(T_1) \cdot V(T_2)}$$

$\left[\because \rho = \dfrac{Cov(x,y)}{\sqrt{V(x) V(y)}} \right]$

$$= \lambda^2 \cdot \frac{V}{e_1} + \mu^2 \frac{V}{e_2} + 2\lambda\mu \cdot \rho \sqrt{\frac{V}{e_1} \cdot \frac{V}{e_2}}$$

(by (1) & (2))

$$V(T_3) = V\left(\frac{\lambda^2}{e_1} + \frac{\mu^2}{e_2} + \frac{2\lambda\mu\rho}{\sqrt{e_1 e_2}}\right)$$

But we know that

$$V(T_3) \geq V$$

$$\Rightarrow V\left[\frac{\lambda^2}{e_1} + \frac{\mu^2}{e_2} + \frac{2\lambda\mu\rho}{\sqrt{e_1 e_2}}\right] \geq \cancel{V}$$

$$\Rightarrow \frac{\lambda^2}{e_1} + \frac{\mu^2}{e_2} + \frac{2\lambda\mu\rho}{\sqrt{e_1e_2}} \geq 1$$

$$\Rightarrow \frac{\lambda^2}{e_1} + \frac{\mu^2}{e_2} + \frac{2\lambda\mu\rho}{\sqrt{e_1e_2}} \geq (\lambda+\mu)^2 \quad (\text{by eq.}^n \ 4)$$

$$\Rightarrow \frac{\lambda^2}{e_1} + \frac{\mu^2}{e_2} + \frac{2\lambda\mu\rho}{\sqrt{e_1e_2}} \geq \lambda^2+\mu^2+2\lambda\mu$$

$$\Rightarrow \left(\frac{1}{e_1}-1\right)\lambda^2 + \left(\frac{1}{e_2}-1\right)\mu^2 + 2\lambda\mu\left(\frac{\rho}{\sqrt{e_1e_2}}-1\right) \geq 0$$

dividing both sides by μ^2, we get

$$\Rightarrow \left(\frac{1}{e_1}-1\right)\frac{\lambda^2}{\mu^2} + \left(\frac{1}{e_2}-1\right) + \frac{2\lambda}{\mu}\left(\frac{\rho}{\sqrt{e_1e_2}}-1\right) \geq 0$$

$$\Rightarrow \left(\frac{1}{e_1}-1\right)\left(\frac{\lambda}{\mu}\right)^2 + 2\left(\frac{\rho}{\sqrt{e_1e_2}}-1\right)\cdot\left(\frac{\lambda}{\mu}\right) + \left(\frac{1}{e_2}-1\right) \geq 0 \quad \rightarrow (5)$$

which is a quadratic eq.^{n} in $\left(\frac{\lambda}{\mu}\right)$.

Since, $e_i < 1 \Rightarrow \frac{1}{e_i} > 1$

$$\Rightarrow \frac{1}{e_i} - 1 > 0 \quad ; \quad i = 1, 2, \ldots$$

We know that $AX^2+BX+C \geq 0 \quad \forall X$

and if $A \geq 0$ & $C \geq 0$ iff discriminent

$B^2 - 4AC \leq 0$

Now from eq.^{n} (5), we have

$$\left[2\left(\frac{\rho}{\sqrt{e_1e_2}}-1\right)\right]^2 - 4\left[\left(\frac{1}{e_1}-1\right)\cdot\left(\frac{1}{e_2}-1\right)\right] \leq 0$$

$$\Rightarrow 4\,\frac{(\rho-\sqrt{e_1e_2})^2}{e_1e_2} - 4\,\frac{(1-e_1)(1-e_2)}{e_1e_2} \leq 0$$

$$\Rightarrow (\rho - \sqrt{e_1 e_2})^2 - (1-e_1)(1-e_2) \le 0$$

$$\Rightarrow \rho^2 - 2\sqrt{e_1e_2}\,\rho + \cancel{e_1e_2} - 1 + e_2 + e_1 - \cancel{e_1e_2} \le 0$$

$$\Rightarrow \rho^2 - 2\sqrt{e_1e_2}\,\rho + (e_1 + e_2 - 1) \le 0.$$

$$\rho = \frac{-(-2\sqrt{e_1e_2}) \pm \sqrt{4e_1e_2 - 4(1)(e_1+e_2-1)}}{2(1)}$$

from Shri Dharacharya formula - $ax^2+bx+c=0$ then $x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$

$$= \frac{2\sqrt{e_1e_2} \pm 2\sqrt{e_1e_2 - e_1 - e_2 + 1}}{2}$$

$$= \sqrt{e_1e_2} \pm \sqrt{e_1(e_2-1) - 1(e_2-1)}$$

$$\rho = \sqrt{e_1e_2} \pm \sqrt{(e_1-1)(e_2-1)}$$

Hence we have

$$\sqrt{e_1e_2} - \sqrt{(e_1-1)(e_2-1)} \le \rho \le \sqrt{e_1e_2} + \sqrt{(e_1-1)(e_2-1)}$$

$$\boxed{\sqrt{e_1e_2} - \sqrt{(1-e_1)(1-e_2)} \le \rho \le \sqrt{e_1e_2} + \sqrt{(1-e_1)(1-e_2)}} \rightarrow (6)$$

H.P.

2005, 2012

Imp. **Theorem - 3** - If T_1 is M.V.U.E. of $\gamma(\theta)$, $\theta \in \Theta$ and T_2 be any other unbiased estimator of $\gamma(\theta)$ with efficiency e then the correlation coefficient between them is given as $\rho = \sqrt{e}$.

Proof:- From eq.^{n} (6) in Theorem 2.

$$\sqrt{e_1e_2} - \sqrt{(1-e_1)\cdot(1-e_2)} \le \rho \le \sqrt{e_1e_2} + \sqrt{(1-e_1)\cdot(1-e_2)}$$

Since the efficiency of M.V.U.E. is 1

i.e. T_1 is M.V.U.E of $\gamma(\theta)$

so, $\boxed{e_1 = 1}$ $\rightarrow$ (7)

and according to que. $\boxed{e_2 = e}$ $\rightarrow$ (8)

Substituting the values of eq.^{n} (7) & (8) in eq.^{n} (6)

$$\sqrt{(1)\cdot e} - \sqrt{(1-1)(1-e_2)} \le \rho \le \sqrt{(1)\cdot e} + \sqrt{(1-1)(1-e)}$$

$\Rightarrow \sqrt{e} \le \rho \le \sqrt{e}$

$\Rightarrow \boxed{\rho = \sqrt{e}}$

H.P.

Imp.

CRAMER - RAO INEQUALITY :-

definition :-

If t is an ~~unb~~ unbiased estimator for $\gamma(\theta)$, a function of parameter θ, then

$$var(t) \geq \frac{\left\{\frac{d}{d\theta}\gamma(\theta)\right\}^2}{E\left(\frac{\partial}{\partial\theta}\log L\right)^2} = \frac{\{\gamma'(\theta)\}^2}{I(\theta)}$$

Where $I(\theta)$ is the information on θ, supplied by the sample.

Most Imp.

UNIT - 2

Interval Estimation :

~~Confidence~~
2004, 2007, 2009, 2011, 2012

* Interval Estimation :-

In point estimation, a single value of a statistic is used as an estimate of the popⁿ.

Thus having obtained a statistic 't', say from a given random sample, the problem arises that :-

"Can we make some reasonable probability statements about the unknown parameter θ in the popⁿ from which the sample has been drawn?" Then the solⁿ is provided by the technique of interval estimation.

This consists in the determination of two constants c_1 & c_2, say such that :

$$\boxed{P[c_1 < \theta < c_2, \text{ for given value of } t] = 1-\alpha \rightarrow (1)}$$

where α is the level of significance.

2004, 2005, 2007, 2008, 2009, 2011, 2012

* Confidence Interval and Confidence Coefficient :-

In the eqⁿ (1) -

The interval $[c_1, c_2]$, within which the unknown value of the parameter θ is expected to lie is known as confidence interval (Neyman) or Fiducial interval (R.A. Fisher).

The limits c_1 and c_2 so determined are known as confidence limits or fiducial limits and $(1-\alpha)$ is called the

Confidence Coefficient, depending upon the desired precision of the estimate.

For example, $\alpha = 0.05$ gives the 95% confidence limits.

The computation of confidence limits for a parameter θ, involves the following steps.

1) Compute the ~~approx~~ appropriate sample statistic 't'.
2) Obtain the standard error of the sampling distⁿ of the statistic 't'. i.e. $S.E(t)$.
3) Choose appropriate confidence coefficient $(1-\alpha)$.

* Interval Estimation for Large Samples:-

For large samples, the underlying distⁿ of the standardized variate corresponding to the sampling distⁿ of the statistic t will be asymptotically normal by Central Limit Theorem.

i.e. $$Z = \frac{t - E(t)}{S.E(t)} \sim N(0,1) \quad ; \; n \to \infty$$

* Confidence Limits and Confidence Interval for Mean (μ):-

If we consider a large random sample of size n from an infinite popⁿ, with mean μ and variance σ^2 then the sample mean $\bar{x} \sim N(\mu, \sigma^2/n)$; $n \to \infty$

i.e. $$Z = \frac{\bar{x} - \mu}{\sigma/\sqrt{n}} \sim N(0,1)$$

For practical purposes, samples may be regarded as large if the sample size is greater then 30 i.e. $n > 30$

From areas under normal probability curve, we have

$$P(-1.96 \le Z \le 1.96) = 0.95 \quad \text{(from normal prob. table)}$$

$$\Rightarrow P\left(-1.96 \le \frac{\bar{x}-\mu}{\sigma/\sqrt{n}} \le 1.96\right) = 0.95$$

$$\Rightarrow P(-1.96\,\sigma/\sqrt{n} \le \bar{x}-\mu \le 1.96\,\sigma/\sqrt{n}) = 0.95$$

$$\Rightarrow P\left(\bar{x} - 1.96\,\sigma/\sqrt{n} \le \mu \le \bar{x} + 1.96\,\sigma/\sqrt{n}\right) = 0.95$$

Thus $\bar{x} \pm 1.96\sigma/\sqrt{n}$ are 95% confidence limit for the unknown parameter μ and the interval $\left[\bar{x} - 1.96\frac{\sigma}{\sqrt{n}}, \bar{x} + 1.96\frac{\sigma}{\sqrt{n}}\right]$ is called the 95% the confidence interval.

* Confidence Limits for Variance :-

In a random sample of size n from an infinite popⁿ with variance σ^2, the variance of the sampling distⁿ of standard deviation (s) is given by

$$\text{Var.}(s) = \frac{\sigma^2}{2n}$$

Hence, $(1-\alpha)$% Confidence limits for the popⁿ S.D. (σ) are given by :-

$$\boxed{s \pm Z_{\alpha/2}\, S.E.(s) = s \pm Z_{\alpha/2}\sqrt{\frac{\sigma^2}{2n}}}$$

2005, 2008, 2009, 2011, 2012, 2013

Q1. Obtain 100 $(1-\alpha)$% confidence interval for the parameter

1) for parameter θ and

2) σ^2 of the normal distⁿ

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \qquad -\infty < x < \infty$$

Solⁿ let x_i $(i=1,2,\ldots n)$ be a random sample of size n from the density function

$$\left.\begin{aligned} \bar{x} &= \frac{1}{n}\sum_{i=1}^{n} x_i \\ s^2 &= \frac{1}{n}\sum_{i=1}^{n}(x_i-\bar{x})^2 \\ S^2 &= \frac{1}{n-1}\sum_{i=1}^{n}(x_i-\bar{x})^2 \end{aligned}\right\} \rightarrow (1)$$

(1) the statistics $t = \dfrac{\bar{x}-\theta}{S/\sqrt{n}}$ follows student 't' distⁿ with $n-1$ degrees of freedom.

Hence 100 $(1-\alpha)$ times confidence limits are given by-

$$P[|t| \le t_\alpha] = 1-\alpha$$

$$\Rightarrow P\left[\left|\frac{\bar{x}-\theta}{S/\sqrt{n}}\right| \le t_\alpha\right] = 1-\alpha$$

$$\Rightarrow P\left[|\bar{x}-\theta| \le t_\alpha \cdot \frac{S}{\sqrt{n}}\right] = 1-\alpha$$

$$\Rightarrow P\left[-t_\alpha\frac{S}{\sqrt{n}} \le \bar{x}-\theta \le t_\alpha\cdot\frac{S}{\sqrt{n}}\right] = 1-\alpha$$

$$\Rightarrow P\left[\bar{x}-t_\alpha\cdot\frac{S}{\sqrt{n}} \le \theta \le \bar{x}+t_\alpha\cdot\frac{S}{\sqrt{n}}\right] = 1-\alpha$$

r the parameter

$< \infty$

ample of size n

$\rightarrow$ ①

't' dist^n

its are

where t_α is the tabulated value of t for $n-1$ degree of freedom at significence level α.

Hence the required confidencial interval for θ is

$$\left(\bar{x} - t_\alpha \cdot \frac{s}{\sqrt{n}}\right), \left(\bar{x} + t_\alpha \cdot \frac{s}{\sqrt{n}}\right)$$

(2) Case - Ist $\Rightarrow$ when θ is known $n = \mu$

Then $\sum_{i=1}^{n} \left(\frac{x_i - \mu}{\sigma}\right)^2 = \frac{n s^2}{\sigma^2} \sim \chi^2_n$ (by eq^n ①)

then the required confidence interval is given by -

$$P\left[\chi^2_{1-\frac{\alpha}{2}} \leq \chi^2 \leq \chi^2_{\alpha/2}\right] = 1 - \alpha$$

$$\Rightarrow P\left[\chi^2_{1-\alpha/2} \leq \frac{n s^2}{\sigma^2} \leq \chi^2_{\alpha/2}\right] = 1 - \alpha$$

Now, $\chi^2_{1-\alpha/2} \leq \frac{n s^2}{\sigma^2} \Rightarrow \sigma^2 \leq \frac{n s^2}{\chi^2_{1-\alpha/2}}$

and $\frac{n s^2}{\sigma^2} \leq \chi^2_{\alpha/2} \Rightarrow \sigma^2 \geq \frac{n s^2}{\chi^2_{\alpha/2}}$

$$\Rightarrow P\left[\frac{n s^2}{\chi^2_{\alpha/2}} \leq \sigma^2 \leq \frac{n s^2}{\chi^2_{1-\alpha/2}}\right] = 1 - \alpha$$

where $\chi^2_{\alpha/2}$ and $\chi^2_{1-\alpha/2}$ are obtained by table by using n d.f.

Case IInd $\Rightarrow$ when θ is unknown

In this case the statistics $\sum_{i=1}^{n} \frac{(x_i - \bar{x})^2}{\sigma^2} = \frac{n s^2}{\sigma^2} \sim \chi^2_{n-1}$

Hence, also confidence interval for σ^2 is given by (2) where now χ^2_α is the significant value of χ^2 for $n-1$ degrees of freedom at the significance level 'α'.

2006, 2007, 2008, 2011, 2012, 2013

Imp

* Definition of Order Statistic :-

Let $X_1, X_2, \ldots, X_n$ be n independent and identically distributed variates are arranged in ascending order of magnitude and then written as $X_{(1)} \leq X_{(2)} \leq \ldots \leq X_{(r)}$, we call $X_{(r)}$ as the r^{th} order statistic $(r = 1, 2, \ldots n)$

If we write these ordered values as-

$Y_1 \leq Y_2 \leq \ldots \leq Y_n$, then

$Y_r = X_r = r^{th}$ smallest of $X_1, X_2, \ldots, X_n$

So, $Y_1 = X_1 =$ The ~~small~~ smallest of $X_1, X_2, \ldots, X_n$

$X_n = X_n =$ The largest of $X_1, X_2, \ldots, X_n$

* Cumulative Distribution Function of a Single Order Statistic :-

Let $F_r(x)$, $r = 1, 2, \ldots n$ denoted the C.d.f (Cumulative Distribution Function) of the r^{th} order statistic $X_{(r)}$. The the C.d.f. $F_n(x)$ of the largest order statistic $X_{(n)}$ is given by -

$$F_n(x) = P[X_{(n)} \leq x] = P[X_i \leq x ; i = 1, 2, \ldots n]$$
$$= P[X_1 \leq x \cap X_2 \leq x \cap \ldots \cap X_n \leq x]$$
$$= P(X_1 \leq x) \cdot P(X_2 \leq x) \cdot \ldots P(X_n \leq x)$$
$$= [F(x)]^n$$

($\because X_i$'s are independent)

$F_r(x) = P[X_{(r)} \leq x]$
$= P[X_1 \leq x \cap$

Since X_1, X_2 ... p.d.f. The C.d.f. $F_1(x)$ ... Since X_1 ... p.d.f. of ... In general ... given by

Since $X_1, X_2, \ldots X_n$ are identically distributed

p.d.f. of $X_{(n)}$ is $\quad f_n(x) = n[F(x)]^{n-1} f(x)$

The c.d.f. $F_1(x)$ of the smallest order statistic $X_{(1)}$ is given by

$$F_1(x) = P[X_{(1)} \le x] = 1 - P[X_{(1)} > x]$$
$$= 1 - P[X_i > x ;\ i = 1,2,\ldots n]$$
$$= 1 - \prod_{i=1}^{n} P(X_i > x)$$
$$= 1 - \prod_{i=1}^{n} [1 - P(X_i \le x)]$$
$$= 1 - [1 - F(x)]^n$$

Since $X_1, X_2, \ldots X_n$ are i.i.d. r.v.'s

p.d.f. of $X_{(1)}$ is $\quad f_1(x) = n[1 - F(x)]^{n-1} \cdot f(x)$

In general the c.d.f. $F_r(x)$ of the r^{th} order statistic $X_{(r)}$ is given by:

$$F_r(x) = P[X_{(r)} \le x] = P[\text{at least } r \text{ of the } x_i\text{'s are} \le x]$$
$$= \sum_{j=r}^{n} P[\text{Exactly } j \text{ of the } n,\ x_i\text{'s are} \le x]$$
$$F_r(x) = \sum_{j=r}^{n} {}^nC_j\, F^j(x)\, [1 - F(x)]^{n-j}$$

by using Binomial probability model.