in general, it may be shown by induction that $S_{2^n} > \left(\frac{n+2}{2}\right)$.

Thus $\{S_n\}_{n=1}^{\infty}$ Contains a divergent subsequence and hence given series diverges.

Theorem 7 :- (Cauchy's general principle of Convergence for series)

A necessary and sufficient condition for a series Σu_n to be convergent is that for every $\varepsilon > 0$, $\exists$ $n_0 \in \mathbb{N}$ such that

$$|S_n - S_m| < \varepsilon$$

i.e., $|u_{m+1} + u_{m+2} + \dots + u_n| < \varepsilon$

$$\forall\ n > m \geq n_0.$$

Proof :- Let $S_n = u_1 + u_2 + \dots + u_n$ be the n^{th} Partial sum of the series Σu_n. Construct the sequence $\langle S_n \rangle$ of it's Partial sums.

Now, $\Sigma u_n \to$ Cong. $\Leftrightarrow$ $\langle S_n \rangle \longrightarrow$ Cong.
[by hypothesis]

Here $S_m = u_1 + u_2 + \dots + u_m$

$S_n = u_1 + u_2 + \dots + u_m + u_{m+1} + \dots + u_n$

$\therefore S_n - S_m = u_{m+1} + u_{m+2} + \dots + u_n$ —— (1)

Theorem 4 :- If $\sum_{n=1}^{\infty} u_n$ is a series of nonnegative numbers with $S_n = u_1 + u_2 + \cdots + u_n$ $(n \in I)$, then

(a) $\sum_{n=1}^{\infty} u_n$ converges if the sequence $\{S_n\}_{n=1}^{\infty}$ is bounded

(b) $\sum_{n=1}^{\infty} u_n$ diverges if $\{S_n\}_{n=1}^{\infty}$ is not bounded.

Theorem 5 :-

(i) If $0 < a < 1$, then $\sum_{n=0}^{\infty} a^n$ converges to $\frac{1}{1-a}$.

(ii) If $a \geq 1$, then $\sum_{n=0}^{\infty} a^n$ diverges.

Theorem 6 :- The series $\sum_{n=1}^{\infty} \left(\frac{1}{n}\right)$ is divergent.

Proof :- We examine the subsequences $S_1, S_2, S_4, S_8, \ldots S_{2^n}, \ldots$ of $\{S_n\}_{n=1}^{\infty}$ where, in this case, $S_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$. We have

$$S_1 = 1$$
$$S_2 = 1 + \frac{1}{2} = 3/2$$
$$S_4 = S_2 + \frac{1}{3} + \frac{1}{4} > \frac{3}{2} + \frac{1}{4} + \frac{1}{4} = 2$$
$$S_8 = S_4 + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} > 2 + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = 5/2$$

Sol^n We find that

$$\frac{1+a}{2+a} - \frac{1}{2} = \frac{a}{2(2+a)} = a \text{ +ve quantity } (\because a>0)$$

$$\Rightarrow \frac{1+a}{2+a} > \frac{1}{4}$$

In general

$$u_n - \frac{1}{2} = \frac{1+(n-1)a}{2+(n-1)a} - \frac{1}{2}$$

$$= \frac{(n-1)a}{2[2+(n-1)a]} \quad (\text{Positive})$$

$$\therefore \quad u_n \geq \frac{1}{3} \quad \forall\, n \in N$$

Hence by theorem 3 the given series is divergent

Example 3 :- Show that the series

$4-4+4-4+ \cdots$ oscillates.

Sol^n :-

Here $S_n = \begin{cases} 4, & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$

$\Rightarrow \lim\limits_{n\to\infty} S_n$ is either 4 or 0

$\Rightarrow \lim\limits_{n\to\infty} S_n$ is not unique.

$\Rightarrow \langle S_n \rangle$ oscillates finitely

$\Rightarrow \Sigma u_n$ oscillates finitely.

Hence the given series oscillates finitely.