$\because \quad 2 < e < 3$

Therefore by ratio test the given series is divergent

Cauchy's n^{th} root test :-

Let $\sum u_n$ be a series of Positive terms such that

$$\lim_{n\to\infty} (u_n)^{1/n} = l$$

(i) if $l < 1$, then $\sum u_n$ is Convergent

(ii) if $l > 1$, then $\sum u_n$ is divergent

(iii) if $l = 1$, $\sum u_n$ may Conv. or diverge.

eg: Let $u_n = \left(\frac{nx}{n+1}\right)^n$

$$\therefore \lim_{n\to\infty} \left\{\left(\frac{nx}{n+1}\right)^n\right\}^{1/n} = \lim_{n\to\infty} \left(\frac{nx}{n+1}\right)$$

$$= \underset{n\to\infty}{Lt} \left(\frac{x}{1+1/n}\right) = x$$

Hence, by Cauchy's n^{th} root test the Series is convergent if $x<1$ and divergent if $x>1$.

If $x=1$, then $u_n = \left(\frac{n}{n+1}\right)^n = \frac{1}{(1+1/n)^n}$

$$\lim_{n\to\infty} u_n = \frac{1}{e} \neq 0$$

$\Rightarrow \sum u_n$ is divergent.

Raabe's test :- Let Σu_n be series of Positive terms such that

$$\lim_{n\to\infty} n\left\{\frac{u_n}{u_{n+1}} - 1\right\} = l$$

then (i) if $l > 1$, Σu_n is Convergent
(ii) if $l < 1$, Σu_n is divergent
(iii) if $l = 1$, the test fails

Two Important limits :-

1. $\lim_{n\to\infty} \frac{\log n}{n} = 0$

2. $\lim_{n\to\infty} (n x^n) = 0$, when $0 < x < 1$

Logarithmic Ratio test :-

Let Σu_n be a Series of Positive terms such that

$$\lim_{n\to\infty} \left[n \log\left(\frac{u_n}{u_{n+1}}\right)\right] = l.$$

then (i) if $l > 1$, Σu_n is Convergent
(ii) if $l < 1$, Σu_n is divergent
(iii) if $l = 1$, Σu_n may Conv. or diverge.

eg: Find whether the series

$$x + x^{1+1/2} + x^{1+1/2+1/3} + x^{1+1/2+1/3+1/4} + \cdots$$ is Convergent or divergent.

Soln :- we have

$$u_n = x^{1+1/2+1/3+\cdots+1/n}$$

$$\therefore \frac{u_n}{u_{n+1}} = \frac{x^{1+1/2+\cdots+1/n}}{x^{1+1/2+\cdots+1/n+1/n+1}} = \frac{1}{x^{1/n+1}}$$

$$n \log\frac{u_n}{u_{n+1}} = n\log\left[\frac{1}{x^{1/n+1}}\right] = n \log\left(1/x\right)^{1/n+1}$$

$$= \frac{n}{n+1}\log\left(\frac{1}{x}\right)$$

$$\lim_{n\to\infty}\left(n\log\frac{u_n}{u_{n+1}}\right) = \lim_{n\to\infty}\frac{n}{n(1+1/n)}\log\left(\frac{1}{x}\right)$$

$$= \log\left(\frac{1}{x}\right)$$

Now, the Series is Convergent if

$\log\left(\frac{1}{x}\right) > 1$ i.e. $\frac{1}{x} > e$ i.e. $x < \frac{1}{e}$

and divergent if $x > 1/e$.

Cauchy's Condensation test :-

If $f(n)$ is a Positive monotonic decreasing function of n, then the Series

$$\Sigma f(n) = f(1) + f(2) + f(3) + \cdots$$

and $\Sigma a^n f(a^n) = af(a) + a^2 f(a^2) + \cdots$

Converge or diverge together, where a is a Positive integer greater than 1.

eg:- Prove that the Series whose n^{th} term is $1/\log n$ is divergent.

Sol^n

Here $f(n) = \frac{1}{\log n}$

Let. $a \in N$

$$\therefore a^n f(a^n) = a^n\cdot\frac{1}{\log a^n} = \frac{a^n}{n\log a}$$

$$= \frac{1}{\log a}\cdot\frac{a^n}{n}$$

but $\sum \frac{a^n}{n}$ is divergent for $\frac{a^n}{n} > \frac{1}{n \log a}$

Hence, the given series is divergent. $(\because a > 1)$

eg: 2 Discuss the nature of the series

$$\frac{1}{2(\log 2)^p} + \frac{1}{3(\log 3)^p} + \frac{1}{4(\log 4)^p} + \dots + \frac{1}{n(\log n)^p} + \dots$$

Solⁿ Let $f(n) = \frac{1}{n(\log n)^p} \quad \forall\, n > 1$

We shall now consider two cases

Case 1

$\boxed{\text{When } p > 0}$

In this case $f(n)$ is a decreasing function of n.

& $a^n f(a^n) = a^n \frac{1}{a^n(\log a^n)^p} = \frac{1}{n^p \cdot (\log a)^p}$

$\Rightarrow \sum a^n f(a^n) = K \cdot \sum \frac{1}{n^p}$ where $K = \frac{1}{(\log a)^p}$

if $p > 1$ then series ~~series~~ convergent — $p < 1$ the series divergent.

But,

By Cauchy's condensation test $\sum f(n)$ and $\sum a^n f(a^n)$ both converge or diverge together. So, the given series $\sum f(n)$ is also convergent if $p > 1$ and divergent if $0 < p \leq 1$.

Case II $\boxed{\text{When } p \leq 0}$

In this case $\frac{1}{n(\log n)^p} \geq \frac{1}{n} \quad \forall\, n > 1$

$\Rightarrow \sum \frac{1}{n(\log n)^p} \geq \sum \frac{1}{n} \rightarrow$ (divergent series)

$\Rightarrow \sum \frac{1}{n(\log p)^n}$ is divergent series

Alternating Series →

A series of the form

$$u_1 - u_2 + u_3 + \cdots + (-1)^{n-1} u_n + \cdots$$

where $u_n > 0 \quad \forall n \in N$

is called an alternating series and it is denoted by $\sum (-1)^{n-1} u_n$.

Leibnitz test :-

The altenating series $u_1 - u_2 + u_3 - u_4 + \cdots$

Converges if (i) $u_n \le u_{n-1} \quad (u_n > 0, \forall n)$

(ii) $u_n \to 0$ as $n \to \infty$

eg: $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$

Sol^n Here the term are altenatively +ve a[nd] −ve and so it is an altenating series

[a]nd $u_n - u_{n-1} = \frac{1}{n} - \frac{1}{n-1}$

$$= \frac{-1}{n(n-1)} \quad (-ve)$$

So, each term is numerically less than the Preceding term

Also, $\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{n} = 0.$

So, by Leibnitz test, the given series is Convergent.

Absolute Convergence :- An infinite series Σu_n of arbitrary terms is said to be absolutely Convergent if the series $\Sigma |u_n|$ of Positive terms is Convergent.

i.e.

Σu_n is abs cgt $\Rightarrow$ $\Sigma |u_n|$ is cgt.

Conditional Convergence :-

Σu_n is Conditionally Congt. $\Rightarrow$ $\Sigma |u_n|$ is divergent

Note :- Every absolute Convergent series is Convergent.

eg: Show that the series

$$\frac{1}{\sqrt{1}} - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \cdots$$

is Conditionally Convergent.

Sol^n The given is alternating series.

Here $u_n = \frac{1}{\sqrt{n}}$, $u_{n+1} = \frac{1}{\sqrt{n+1}}$

$\therefore$ $u_n - u_{n+1} = \frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}} > 0 \quad \forall n \in \mathbb{N}$

i.e., $u_n > u_{n+1}$

Also, $\lim_{n\to\infty} u_n = \lim_{n\to\infty}\left(\frac{1}{\sqrt{n}}\right) = 0$

Hence by Leibnitz' test $\Sigma \frac{(-1)^{n-1}}{\sqrt{n}}$ is Convergent

Exercise

Test for Convergence the following series.

1. $1 + \frac{3}{2} + \frac{5}{2^2} + \frac{7}{2^3} + \cdots$

2. $\frac{x}{1.2} + \frac{x^2}{2.3} + \frac{x^3}{3.4} + \cdots$ for $x > 0$

3. $\frac{1}{2} + \frac{1}{3^2} + \frac{1}{2^3} + \frac{1}{3^4} + \cdots$

4. $1 + \frac{1}{2}x + \frac{1.3}{2.4}x^3 + \frac{1.3.5}{2.4.6}x^4 + \cdots$

5. $\sum \frac{1.3.5 \ldots (2n-1)}{2.4.6 \ldots 2n} \cdot \frac{1}{n}$

6. $\sum \frac{n!}{n^n} \cdot x^n$

7. $(1)^p + \left(\frac{1}{2}\right)^p + \left(\frac{1.3}{2.4}\right)^p + \left(\frac{1.3.5}{2.4.6}\right)^p + \cdots, \; P >$

8. $1^a.2^b + 2^a.3^b + 3^a.4^b + \cdots$

9. $\sum (-1)^n \left(\sqrt{(n^2+1)} - n \right]$

10. $\frac{1}{a} - \frac{1}{a+x} + \frac{1}{a+2x} - \frac{1}{a+3x} + \cdots$

11. $1 - 2 + 3 - 4 + 5 - 6 + \cdots$

12. $\frac{1}{1.2} - \frac{1}{3.4} + \frac{1}{5.6} - \frac{1}{7.8} + \cdots$

13. $\sum (-1)^n \sin \frac{1}{n}$.