THIRD YEAR B.Sc. MATHEMATICS 2017-18

PAPER- III (B)
NUMERICAL ANALYSIS AND OPERATIONS RESERCH

Duration: 3 Hours **Max. Marks: 75**

UNIT - I

Differences, Relation between differences and derivatives, differences of Polynomial, Newton-Gregory formula for forward and backward interpolation, divided differences. Newton's General interpolation formula, Lagranges's interpolation formula.

UNIT - II

Gauss's central difference formula, Stirling's and Bessels interpolation formula, Inverse interpolation. Numerical differentiation, Derivatives from Interpolation formulae. Method of operators, Numerical Integration: Newton-cotes Quadrature formula, Trapezoidal, Simpson's one third, Simpson's three-eight rules.

UNIT-III

Gauss Quadrature formulae, Estimation of errors in quadrature formula, location of roots by Descarte's method of sign, Newtons theorem on multiple roots, Numerical solution of Algebraic and Transcendental equations, Bisection method, Regula-Falsi method, Method of integration .

UNIT-IV

Introduction to linear programming problems, Mathematical formulation Graphical method of solution of linear programming problems (Problems of two variables only), Theory of convex sets, Theory of Simplex method and its applications to simple linear programming problems.

UNIT - V

Concepts of duality in linear programming, formation of dual problems, Elementary theorems of duality. Assignment and transportation problems and their optimum solutions.

References:

1. C. E. Froberg : Introduction to Numerical Analysis
2. M. K. Jain, S. R. K. Iyenger and R.K. Jain : Numerical methods: Problems & solutions
3. G. Hadley : Linear Programming
4. Kanti Swaroop, P. K. Gupta and Man Mohan : Operation Research
5. H.C. Saxena : Numerical Analysis
6. Goyal, Mittal : Numerical Analysis
7. Goyal, Mittal : Numerical Analysis (Hindi ed.)
8. Goyal, Mittal : Numerical Analysis (Hindi ed.)
9. Goyal, Mittal : Operations Research
10. S.D.Sharma : Operations Research
11. Gokhroo, Saini, Jain : Operations Research (Hindi ed.)
12. Bhargava, Bhati, Sharma : Linear Programming (Hindi ed.)
13. Gokhroo, Saini, Jain : Linear Programming (Hindi ed.)

B.Sc. IIIrd Year Mathematics

Paper – III (B) Numerical Analysis & Operations Research

Introduction

Numerical Analysis: Numerical Analysis is a technique to find the numerical values of physical problem, which could be in the form of differential equations or algebraic equations. In most cases, it can not be solved analytically therefore it can be calculated using a finite number of basic arithmetic operations.

Hence, a major advantage for numerical analysis is that a numerical answer can be obtained even when a problem has no analytical solution. However, result from numerical analysis is an approximation.

Operations Research: Operations research (OR) is a scientific method of providing executive with an analytical and objective basis for decisions.

OR can be associated with "an art of winning the war without actually fighting it".

OR is useful in the following various important fields.

1. Agriculture
2. Finance
3. Industry
4. Marketing
5. Personal Management
6. Production Management
7. L.I.C.
8. Research and Development

Unit - I The calculus of finite Differences

Forward Difference: Let $y = f(x)$ be a function and for the consecutive values of x i.e. $a, a+h, a+2h, \ldots, a+nh$ corresponding values of $f(x)$ are $f(a), f(a+h), f(a+2h), \ldots, f(a+nh)$.

The difference of $f(a)$ from next value $f(a+h)$ i.e. $f(a+h) - f(a)$ is called first forward difference denoted by $\Delta f(a)$

i.e. $\Delta f(a) = f(a+h) - f(a)$

where Δ, is called forward difference operation

similarly $\Delta f(a+h) = f(a+2h) - f(a+h)$

$\Delta f(a+2h) = f(a+3h) - f(a+2h)$

in general first forward difference operation defined as

$$\boxed{\Delta f(x) = f(x+h) - f(x)}$$

Second forward difference

$$\Delta^2 f(x) = \Delta \cdot [\Delta f(x)]$$

$$= \Delta [f(x+h) - f(x)]$$

$$= \Delta f(x+h) - \Delta f(x)$$

$$= f(x+2h) - f(x+h) - [f(x+h) - f(x)]$$

$$\boxed{\Delta^2 f(x) = f(x+2h) - 2f(x+h) + f(x)}$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$\Delta^n f(x) = \Delta \cdot \Delta \cdot \Delta \cdots \Delta (n \text{ times})\, f(x)$$

Forward Difference table

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$
a	$f(a)$		
		$\cancel{\Delta f(a)}$ $\Delta f(a) = f(a+h) - f(a)$	
$a+h$	$f(a+h)$		$\Delta^2 f(a) = \Delta f(a+h) - \Delta f(a)$
		$\Delta f(a+h) = f(a+2h) - f(a+h)$	
$a+2h$	$f(a+2h)$		$\Delta^2 f(a+h) = \Delta f(a+2h) - \Delta f(a+h)$
		$\Delta f(a+2h) = f(a+3h) - f(a+2h)$	
$a+3h$	$f(a+3h)$		

$\Delta^3 f(x)$

$$\Delta^3 f(a) = \Delta^2 f(a+h) - \Delta^2 f(a)$$

See the example on Next page.

Q: If $f(-1) = -13$, $f(0) = -7$, $f(1) = -1$, $f(2) = 11$, $f(3) = 35$, then find the value of $f(4) = ?$ also if $f(x) = x^3 + 5x - 7$ then verify the values.

Sol^n: given that x : $-1, 0, 1, 2, 3$ at equal interval

and $f(x)$: $-13, -7, -1, 11, 35$

To find $f(4)$ construct forward Difference table

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
-1	-13			
		6		
0	-7		0	
		6		6
1	-1		6	
		12		6
2	11		12	
		24		$y - 71 = 6 \Rightarrow y = 77$
3	35		$y - 59$	
		$y - 35$		
4	y			

verify

$f(4) = 4^3 + 5 \times 4 - 7$

$= 64 + 20 - 7$

$= 77$

Finding any value of the function $f(x)$ in leading terms

i.e. $f(a+nh) = f(a) + {}^nc_1 \Delta f(a) + {}^nc_2 \Delta^2 f(a) + \cdots + \Delta^n f(a)$

<u>proof:-</u> $\because \Delta f(a) = f(a+h) - f(a)$

$\Rightarrow f(a+h) = f(a) + \Delta f(a)$ ——— (1)

Now $a \to a+h$ in (1) we get

$f(a+2h) = f(a+h) + \Delta f(a+h)$

$= f(a) + \Delta f(a) + \Delta[f(a) + \Delta f(a)]$

$\Rightarrow f(a+2h) = f(a) + 2\Delta f(a) + \Delta^2 f(a)$ ——— (2)

$= f(a) + 2c_1 \Delta f(a) + \Delta^2 f(a)$

again $a \to a+2h$ in (1) we get

$f(a+3h) = f(a+2h) + \Delta f(a+2h)$

$= f(a) + 2\Delta f(a) + \Delta^2 f(a)$

$+ \Delta[f(a) + 2\Delta f(a) + \Delta^2 f(a)]$ by (2)

$\Rightarrow f(a+3h) = f(a) + 3\Delta f(a) + 3\Delta^2 f(a) + \Delta^3 f(a)$ ——— (3)

$= f(a) + 3c_1 \Delta f(a) + 3c_2 \Delta^2 f(a) + \Delta^3 f(a)$

again $a \to a+3h$ in (1) we get

$f(a+4h) = f(a+3h) + \Delta f(a+3h)$

$= f(a) + 3\Delta f(a) + 3\Delta^2 f(a) + \Delta^3 f(a)$

$+ \Delta[f(a) + 3\Delta f(a) + 3\Delta^2 f(a) + \Delta^3 f(a)]$

$\Rightarrow f(a+4h) = f(a) + 4\Delta f(a) + 6\Delta^2 f(a) + 4\Delta^3 f(a) + \Delta^4 f(a)$ ——— (4)

$= f(a) + 4c_1 \Delta f(a) + 4c_2 \Delta^2 \Delta f(a) + 4c_3 \Delta^3 f(a) + \Delta^4 f(a)$

by eq^n (1), (2), (3) & (4) we have

$$f(a+h) = f(a) + \Delta f(a)$$

$$f(a+2h) = f(a) + {}^2c_1 \Delta f(a) + \Delta^2 f(a)$$

$$f(a+3h) = f(a) + {}^3c_1 \Delta f(a) + {}^3c_2 \Delta^2 f(a) + \Delta^3 f(a)$$

$$f(a+4h) = f(a) + {}^4c_1 \Delta f(a) + {}^4c_2 \Delta^2 f(a) + {}^4c_3 \Delta^3 f(a) + \Delta^4 f(a)$$

Hence by Mathematical induction method

$$f(a+nh) = f(a) + {}^nc_1 \Delta f(a) + {}^nc_2 \Delta^2 f(a) + \text{---} + \Delta^n f(a)$$

Q: Find the population of a city in 1961 from the given table:

Year	1901	1911	1921	1931
Population (In Lakhs)	46	66	81	93

Sol^n: To find the population of a city in 1961 from the given table, we construct the forward difference table ie

Year (x)	Population $f(x)$ (in Lakh)	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
1901	$\boxed{46}$			
		$\boxed{20}$		
1911	66		$\boxed{-5}$	
		15		$\boxed{2}$
1921	81		-3	
		12		
1931	93			

use the formula

$$f(a+nh) = f(a) + n_{c_1} \Delta f(a) + n_{c_2} \Delta^2 f(a) + \text{———} + \Delta^n f(a)$$

where $a = 1901, \ h = 10$

$a+nh = 1961$

$\Rightarrow 1901 + n \times 10 = 1961$

$\Rightarrow n = \dfrac{1961 - 1901}{10}$

$\boxed{n = 6}$

$\therefore f(1961) = 46 + 6_{c_1} \times 20 + 6_{c_2} \times (-5) + 6_{c_3} \times (2)$

$= 46 + 6 \times 20 + 15 \times (-5) + 20 \times 2$

$= 46 + 120 - 75 + 40$

$= 131$

Hence the population of the city in year 1961 is Approximate 131 Lakh.

Q: If $\sin 30^\circ = 0.5000$, $\sin 35^\circ = 0.5736$, $\sin 40^\circ = 0.6428$ and $\sin 45^\circ = 0.7071$ then find the value of $\sin 90^\circ$.

Sol. To find the value of $\sin 90^\circ$ from the given values

$\sin 30^\circ = 0.5000$

$\sin 35^\circ = 0.5736$

$\sin 40^\circ = 0.6428$

$\sin 45^\circ = 0.7071$

let $f(x) = \sin x^\circ$

and construct the forward Difference table

x°	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
30	0.5000			
		0.0736		
35	0.5736 ~~0.75~~		-0.0044	
		0.0692		-0.0005
40	0.6428		-0.0049	
		0.0643		
45	0.7071			

use $f(a+nh) = f(a) + {}^{n}C_1 \Delta f(a) + {}^{n}C_2 \Delta^2 f(a) + \cdots + \Delta^n f(a)$

where $a = 30$, $h = 5$, $a+nh = 90$

$$\Rightarrow 30 + n \times 5 = 90$$

$$\Rightarrow n = \frac{90-30}{5}$$

$$\Rightarrow \boxed{n = 12}$$

$$f(90) = 0.5000 + {}^{12}C_1 \times 0.0736 + {}^{12}C_2 \times (-0.0044) + {}^{12}C_3 \times (-0.0005)$$

$$= 0.5000 + 12 \times 0.0736 - 66 \times 0.0044 - 220 \times 0.0005$$

$$= 0.5000 + 0.8832 - 0.2904 - 0.11$$

$$= 0.9829 \cong 1$$

Hence $\boxed{\sin 90^\circ \cong 1}$

Newton-Gregory forward interpolation formulas:

$$f(x) = f(a) + u\Delta f(a) + \frac{u(u-1)}{2!}\Delta^2 f(a) + \frac{u(u-1)(u-2)}{3!}\Delta^3 f(a) + \text{---}$$

where $u = \frac{x-a}{h}$

Q: Find $f(15)$ from the given table

x :	3	5	7	9	11
$f(x)$:	6	24	58	108	174

also find $f(6)$ & $f(x)$.

Sol: To find $f(6)$, $f(15)$ & $f(x)$ from the given table

x :	3	5	7	9	11
$f(x)$:	6	24	58	108	174

construct difference table

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$
3	6		
		18	
			16
5	24		
		34	
			16
7	58		
		50	
			16
9	108		
		66	
11	174		

for $f(15)$ use the formula.

$$f(a+nh) = f(a) + {}^nC_1\,\Delta f(a) + {}^nC_2\,\Delta^2 f(a) + \text{---}$$

when $a = 3$, $h = 2$, $a + nh = 15$

$$n = \frac{15-3}{2} = 6$$

$$f(15) = 6 + {}^6C_1 \times 18 + {}^6C_2 \times 16$$

$$= 6 + 6\times 18 + \frac{6\times 5}{2}\times 16 = 6 + 108 + 240 = 354$$

for $f(6)$ use the formula:

$$f(x) = f(a) + u\Delta f(a) + \frac{u(u-1)}{2!}\Delta^2 f(a) + \frac{u(u-1)(u-2)}{3!}\Delta^3 f(a) + \ldots$$

where $u = \frac{x-a}{h} = \frac{6-3}{2} = 1.5$

$$f(6) = 6 + 1.5 \times 18 + \frac{1.5 \times 0.5}{2} \times 16$$

$$= 6 + 27 + 6$$

$$\boxed{f(6) = 39}$$

for $f(x)$, $u = \frac{x-3}{2}$

$$f(x) = 6 + \left(\frac{x-3}{2}\right) \times 18 + \frac{\left(\frac{x-3}{2}\right)\left[\frac{x-3}{2} - 1\right]}{2} \times 16$$

$$= 6 + 9(x-3) + 2(x-3)(x-5)$$

$$= 6 + 9x - 27 + 2(x^2 - 8x + 15)$$

$$= 2x^2 - 7x + 9$$

verify

$$f(6) = 2 \times (6)^2 - 7 \times 6 + 9$$

$$= 2 \times 36 - 42 + 9$$

$$= 39$$

$$f(15) = 2 \times 15^2 - 7 \times 15 + 9$$

$$= 450 - 105 + 9$$

$$= 354$$

Q.11 In a examination the number of candidates who obtained marks between certain limits are as follows

Marks :	0 - 19	20 - 39	40 - 59	60 - 79	80 - 99
No of candidates :	41	62	65	50	17

Estimate the number of candidates who obtained less than 70 marks.

Sol: Given data's as follows

Marks :	0 - 19	20 - 39	40 - 59	60 - 79	80 - 99
No of Candidates :	41	62	65	50	17

To find number of candidates who obtained less than 70 marks, construct the difference table of less than marks.

Marks (x) less than	No of cand' $f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
20	$\boxed{41}$			
		$\boxed{62}$		
40	103		$\boxed{3}$	
		65		$\boxed{-18}$
60	168		-15	
		50		-18
80	218		-33	
		17		
100	235			

use Newton Gregory forward difference interpolation formula

$$f(x) = f(a) + u\Delta f(a) + \frac{u(u-1)}{2!}\Delta^2 f(a) + \frac{u(u-1)(u-2)}{3!}\Delta^3 f(a)$$

$+$ ——

$$u = \frac{x-a}{h}$$

when $x = 70,\ a = 20,\ h = 20 \Rightarrow u = \frac{5}{2}$

$$f(70) = 41 + \frac{5}{2} \times 62 + \frac{\frac{5}{2} \times \frac{3}{2}}{2} \times 3 + \frac{\frac{5}{2} \times \frac{3}{2} \times \frac{1}{2}}{6} \times (-18)$$

$$= 41 + 155 + 5.625 - 5.625$$

$$= 196$$

Hence the number of candidates who obtained less than 70 marks are 196.

Q.2 Use Newton's forward difference formula to obtain the polynomial $f(x)$ satisfying the following data,

x :	1	2	3	4
$f(x)$:	26	18	4	1

If another data $x = 5$, $f(x) = 26$ is added to the above ~~ata~~ data, will the polynomial be the same as before or different? Explain why.

Sol: Given data's are

x :	1	2	3	4
$f(x)$:	26	18	4	1

To find polynomial construct the difference table

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
1	26			
		-8		
2	18		-6	
		-14		17
3	4		11	
		-3		
4	1			

Newton's forward difference formula

$$f(x) = f(a) + u\Delta f(a) + \frac{u(u-1)}{2!}\Delta^2 f(a) + \frac{u(u-1)(u-2)}{3!}\Delta^3 f(a) + \text{---}$$

when $a = 1$, $h = 1$, $u = \frac{x-1}{1} = x-1$

$$f(x) = 26 + (x-1)\times(-8) + \frac{(x-1)(x-2)}{2}\times(-6)$$

$$+ \frac{(x-1)(x-2)(x-3)}{6}\times 17$$

$$= 26 - 8(x-1) - 3(x^2 - 3x + 2) + \frac{17}{6}(x^2-3x+2)(x-3)$$

$$= 26 - 8x + 8 - 3x^2 + 9x - 6 + \frac{17}{6}\left[x^3 - 6x^2 + 11x - 6\right]$$

$$\boxed{f(x) = \frac{17}{6}x^3 - 20x^2 + \frac{193}{6}x + 11}$$

If another data $x = 5$, $f(x) = 26$, is added then differ table

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
1	26			
		-8		
			-6	
2	18			17
		-14		
			11	
3	4			17
		-3		
			28	
4.	1			
		25		
5	26			

then polynomial become same as before because $\Delta^3 f(x)$ are same ∴ therefore $\Delta^4 f(x)$ become zero.

Backward Interpolation formula

$\nabla f(x) = f(x) - f(x-h)$

Backward Difference table

x	$f(x)$	$\nabla f(x)$	$\nabla^2 f(x)$
a	$f(a)$		
		$f(a+h)-f(a) = \nabla f(a+h)$	
$a+h$	$f(a+h)$		$\nabla^2 f(a+2h) = \nabla f(a+2h) - \nabla f(a+h)$
		$f(a+2h) - f(a+h) = \nabla f(a+2h)$	
$a+2h$	$f(a+2h)$		

Backward Interpolation formular (Newton-Gregory)

$$f(x) = f(a+nh) + u\,\nabla f(a+nh) + \frac{u(u+1)}{2!}\nabla^2 f(a+nh) + \frac{u(u+1)(u+2)}{3!}\nabla^3 f(a+nh) + \text{---}$$

$$u = \frac{x-(a+nh)}{h}$$

Q.1 Find $f(7.5)$ from the given data's

x :	1	2	3	4	5	6	7	8
$f(x)$:	1	8	27	64	125	216	343	512

Sol^n. To find $f(7.5)$ from the given data

x :	1	2	3	4	5	6	7	8
$f(x)$:	1	8	27	64	125	216	343	512

use Newton-Gregory Backward difference interpolation formula

$$f(x) = f(a+nh) + u\,\nabla f(a+nh) + \frac{u(u+1)}{2!}\nabla^2 f(a+nh)$$

$$+ \frac{u(u+1)(u+2)}{3!}\nabla^3 f(a+nh) + \text{---}$$

where $u = \frac{x-(a+nh)}{h}$, $a+nh = 8$, $h = 1$, $x = 7.5$

$$u = \frac{7.5-8}{1} \Rightarrow \boxed{u = -0.5}$$

for this construct difference table

x	$f(x)$	$\nabla f(x)$	$\nabla^2 f(x)$	$\nabla^3 f(x)$
1	1			
		7		
			12	
2	8			6
		19		
			18	
3	27			
		37		6
			24	
4	64			
		61		6
			30	
5	125			
		91		6
			36	
6	216			
		127		6
			42	
7	343			
		169		
8	512			

$$f(7.5) = 512 + \left(-\tfrac{1}{2}\right)\times 169 + \frac{\left(-\tfrac{1}{2}\right)\times\left(\tfrac{1}{2}\right)\times 42}{2} + \frac{\left(-\tfrac{1}{2}\right)\times\left(\tfrac{1}{2}\right)\left(\tfrac{3}{2}\right)\times 6}{6}$$

$$= 512 - 84.5 - 5.25 - 0.375$$

$$= 421.875 \text{ (Appo)}$$

New $(7.5)^3 = 421.875$

Q: Apply Newton-Gregory backword formula to the following table to find sun's declination on Feb 12 :

Date :	1	3	5	7	9	11	13
Declination :	$-17^\circ 0' 19.0''$	$-16^\circ 25' 22.9''$	$-15^\circ 49' 18.8''$	$-15^\circ 12' 09.8''$	$-14^\circ 33' 59.1''$	$-13^\circ 54' 49.8''$	$-13^\circ 14' 45.0''$

<u>Solution</u> : To use Newton-Gregory backward formula for finding declination of feb 12 from the given table construct difference table

Date (x)	Declination $f(x)$	$\nabla f(x)$	$\nabla^2 f(x)$	$\nabla^3 f(x)$	$\nabla^4 f(x)$
1	$-17^\circ 0' 19.0''$				
		$0^\circ 34' 56.1''$			
3	$-16^\circ 25' 22.9''$		$0^\circ 1' 8''$		
		$0^\circ 36' 4.1''$		$-0^\circ 0' 3.1''$	
			$0^\circ 1' 4.9''$		$-0^\circ 0' 0.1''$
5	$-15^\circ 49' 18.8''$			$-0^\circ 0' 3.2''$	
		$0^\circ 37' 9''$	$0^\circ 1' 1.7''$		$0^\circ 0' 0.1''$
7	$-15^\circ 12' 09.8''$	$0^\circ 38' 10.7''$		$-0^\circ 0' 3.1''$	$0^\circ 0' 0''$
9	$-14^\circ 33' 59.1''$		$0^\circ 0' 58.6''$	$-0^\circ 0' 3.1''$	
		$0^\circ 39' 9.3''$			
11	$-13^\circ 54' 49.8''$		$0^\circ 0' 55.5''$		
		$0^\circ 40' 4.8''$			
13	$-13^\circ 14' 45.0''$				

$\nabla^5 f(x)$	$\nabla^6 f(x)$
$0^\circ 0' 0.2''$	
	$-0^\circ 0' 0.3''$
$-0^\circ 0' 0.1''$	

Explanation

x	$f(x)$	$\Delta f(x)$
1	$-17^\circ 0' 19.0''$	$-16^\circ 25' 22.9'' - (-17^\circ 0' 19.0'') = 0^\circ 34' 56.1''$
3	$-16^\circ 25' 22.9''$	

i.e. $\begin{array}{r} 17^\circ 0' 19.0'' \\ -16^\circ 25' 22.9'' \end{array} \Rightarrow \begin{array}{r} 16^\circ 59' 79.0'' \\ -16^\circ 25' 22.9'' \\ \hline 0^\circ 34' 56.1'' \end{array}$

similarly all (or using scientific calculator)

for Feb 12: $x = 12$, $a+nh = 13$, $h = 2$

Newton Gregory backward interpolation formula

$$f(x) = f(a+nh) + u\nabla f(a+nh) + \frac{u(u+1)}{2!}\nabla^2 f(a+nh) + \frac{u(u+1)(u+2)}{3!}\nabla^3 f(a+nh) + \text{---} -$$

where $u = \dfrac{x-(a+nh)}{h} = \dfrac{12-13}{2} = -\dfrac{1}{2}$

$$f(12) = -13^\circ 14' 45.0'' + \left(-\tfrac{1}{2}\right) \times 0^\circ 40' 4.8'' + \frac{\left(-\tfrac{1}{2}\right) \times \tfrac{1}{2} \times 0^\circ 0' 55.5''}{2}$$

$$+ \frac{\left(-\tfrac{1}{2}\right) \times \tfrac{1}{2} \times \tfrac{3}{2} \times (-0^\circ 0' 3.1'')}{6} + \frac{\left(-\tfrac{1}{2}\right) \times \tfrac{1}{2} \times \tfrac{3}{2} \times \tfrac{5}{2} \times (0^\circ 0' 0'')}{24}$$

$$+ \frac{\left(-\tfrac{1}{2}\right) \times \tfrac{1}{2} \times \tfrac{3}{2} \times \tfrac{5}{2} \times \tfrac{7}{2} \times (-0^\circ 0' 0.1'')}{120} + \frac{\left(-\tfrac{1}{2}\right) \times \tfrac{1}{2} \times \tfrac{3}{2} \times \tfrac{5}{2} \times \tfrac{7}{2} \times \tfrac{9}{2} \times (-0^\circ 0' 0.3'')}{720}$$

$$= -13^\circ 14' 45.0'' - 0^\circ 20' 2.4'' - 0^\circ 0' 6.94'' + 0^\circ 0' 0.19''$$

$$-0 + 0^\circ 0' 0.01'' + 0^\circ 0' 0.01''$$

$f(12) = -13^\circ 34' 54.13''$

Hence sun's declination on Feb 12 is $-13^\circ 34' 54.13''$ (Approx).

Lagrange's Interpolation formula for unequal interval

x:	x_0	x_1	x_2	x_3	---	x_n
$f(x)$:	$f(x_0)$	$f(x_1)$	$f(x_2)$	$f(x_3)$	---	$f(x_n)$

$$f(x) = \frac{(x-x_1)(x-x_2)\cdots(x-x_n)}{(x_0-x_1)(x_0-x_2)\cdots(x_0-x_n)} f(x_0)$$

$$+ \frac{(x-x_0)(x-x_2)\cdots(x-x_n)}{(x_1-x_0)(x_1-x_2)\cdots(x_1-x_n)} f(x_1)$$

$$+ \cdots$$

$$+ \frac{(x-x_0)(x-x_1)\cdots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\cdots(x_n-x_{n-1})} f(x_n)$$

Q1 Find the value of y when $x = 10$ from the following data's

x :	5	6	9	11
y :	12	13	14	16

Sol Given data's

	x_0	x_1	x_2	x_3
x :	5	6	9	11
y :	12	13	14	16
	y_0	y_1	y_2	y_3

to find y at $x = 10$ use Lagrange's interpolation formula

$$y = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} y_1$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} y_3$$

$$= \frac{(10-6)(10-9)(10-11)}{(5-6)(5-9)(5-11)} \times 12 + \frac{(10-5)(10-9)(10-11)}{(6-5)(6-9)(6-11)} \times 13$$

$$+ \frac{(10-5)(10-6)(10-11)}{(9-5)(9-6)(9-11)} \times 14 + \frac{(10-5)(10-6)(10-9)}{(11-5)(11-6)(11-9)} \times 16$$

$$y = \frac{4\times1\times(-1)}{(-1)\times(-4)\times(-6)}\times12 + \frac{5\times1\times(-1)}{1\times(-3)\times(-5)}\times13$$

$$+ \frac{5\times4\times(-1)}{4\times3\times(-2)}\times14 + \frac{5\times4\times1}{6\times5\times2}\times16$$

$$= 2 - \frac{13}{3} + \frac{35}{3} + \frac{16}{3}$$

$$= 2 + 12.6667$$

$$y = 14.6667 \text{ (Approx)}$$

Q2 Using Lagrange's interpolation formula find $y(3)$ from the following table

x :	0	1	2	4	5
y :	0	16	48	88	0

Sol^n Given table

	x_0	x_1	x_2	x_3	x_4
x :	0	1	2	4	5
y :	0	16	48	88	0
	y_0	y_1	y_2	y_3	y_4

Lagrange's formula

$$y = \frac{(x-x_1)(x-x_2)(x-x_3)(x-x_4)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)(x_0-x_4)}\times y_0$$

$$+ \frac{(x-x_0)(x-x_2)(x-x_3)(x-x_4)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)(x-x_4)}\times y_1$$

$$+ \text{-----}$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_2)(x-x_3)}{(x_4-x_0)(x_4-x_1)(x_4-x_2)(x_4-x_3)}\times y_4$$

for $x = 3$

$$y = \frac{(3-1)(3-2)(3-4)(3-5)}{(0-1)(0-2)(0-4)(0-5)} \times 0 + \frac{(3-0)(3-2)(3-4)(3-5)}{(1-0)(1-2)(1-4)(1-5)} \times 16$$

$$+ \frac{(3-0)(3-1)(3-4)(3-5)}{(2-0)(2-1)(2-4)(2-5)} \times 48 + \frac{(3-0)(3-1)(3-2)(3-5)}{(4-0)(4-1)(4-2)(4-5)} \times 88$$

$$+ \frac{(3-0)(3-1)(3-2)(3-4)}{(5-0)(5-1)(5-2)(5-4)} \times 0$$

$$= 0 + \frac{3 \times 1 \times (-1) \times (-2)}{1 \times (-1) \times (-3) \times (-4)} \times 16 + \frac{3 \times 2 \times (-1) \times (-2)}{2 \times 1 \times (-2) \times (-3)} \times 48$$

$$+ \frac{3 \times 2 \times 1 \times (-2)}{4 \times 3 \times 2 \times (-1)} \times 88 + 0$$

$$= -8 + 48 + 44$$

$$y(3) = 84$$

Divided Difference

First Divided Difference

$$f(x_0, x_1) = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \underset{x_1}{\Delta} f(x_0)$$

Second Divided Difference

$$f(x_0, x_1, x_2) = \underset{x_1, x_2}{\Delta^2} f(x_0) = \frac{f(x_1, x_2) - f(x_0, x_1)}{x_2 - x_0}$$

third divided difference

$$f(x_0, x_1, x_2, x_3) = \frac{f(x_1, x_2, x_3) - f(x_0, x_1, x_2)}{x_3 - x_0}$$

$$\vdots \qquad \vdots \qquad \vdots$$

n^{th} divided difference

$$f(x_0, x_1, \ldots x_n) = \frac{f(x_1, x_2 \ldots x_n) - f(x_0, x_1 \ldots x_{n-1})}{x_n - x_0}$$

$$= \underset{x_1, x_2, \ldots x_n}{\Delta^n} f(x_0)$$

Divided Difference Table

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
x_0	$f(x_0)$			
		$f(x_0, x_1)$		
x_1	$f(x_1)$		$f(x_0, x_1, x_2)$	
		$f(x_1, x_2)$		$f(x_0, x_1, x_2, x_3)$
x_2	$f(x_2)$		$f(x_1, x_2, x_3)$	
		$f(x_2, x_3)$		
x_3	$f(x_3)$			

Newton's Divided Difference interpolation formula for unequal intervals.

$$f(x) = f(x_0) + (x-x_0) f(x_0, x_1) + (x-x_0)(x-x_1) f(x_0, x_1, x_2)$$

$$+ \text{———} \quad \text{———}$$

$$+ (x-x_0)(x-x_1) \text{———} (x-x_{n-1}) f(x_0, x_1, \text{———} x_n)$$

Q. Construct the divided difference table and find the value of $f(2)$ from $f(0) = 8$, $f(1) = 68$, $f(5) = 123$

Sol: Given data's $f(0) = 8$, $f(1) = 68$, $f(5) = 123$

Divided Difference table

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$
0	8		
		60	
1	68		-9.25
		13.75	
5	123		

for $f(2)$ use Newton's divided Difference formula

$$f(x) = f(x_0) + (x-x_0)\Delta f(x) + (x-x_0)(x-x_1)\Delta^2 f(x)$$

$$f(2) = 8 + 2 \times 60 + 2 \times 1 \times (-9.25)$$

$$= 8 + 120 - 18.5 = 109.5$$

Q. Find the polynomial for the following table, hence find $f(10)$.

x:	0	2	5	9	11
$f(x)$:	1	5	116	712	1310

Sol. To find the polynomial for the table

x :	0	2	5	9	11
$f(x)$:	1	5	116	712	1310

∵ values of x are in unequal interval

∴ we use newton's divided difference formula.

$$f(x) = f(x_0) + (x-x_0)\Delta f(x_0) + (x-x_0)(x-x_1)\Delta^2 f(x_0)$$
$$+ (x-x_0)(x-x_1)(x-x_2)\Delta^3 f(x_0) + \text{———}$$

for the difference table

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
0	1				
		2			
2	5		7		
		37		1	
5	116		16		0
		149		1	
9	712		25		
		299			
11	1310				

$$f(x) = 1 + x \times 2 + x.(x-2) \times 7 + x.(x-2).(x-5).1$$

$$= 1 + 2x + 7(x^2-2x) + x.(x^2-7x+10)$$

$$= 1 + 2x + 7x^2 - 14x + x^3 - 7x^2 + 10x$$

$$f(x) = x^3 - 2x + 1$$

Now
$$f(10) = 10^3 - 2 \times 10 + 1$$
$$= 981$$

Q: find $f(301)$ from the table

x :	300	304	305	307
$f(x)$:	2.4771	2.4829	2.4843	2.4871

Sol: To find $f(301)$ from the given table we use Newton's divided difference interpolation formula

i.e. $f(x) = f(x_0) + (x-x_0)\Delta f(x_0) + (x-x_0)(x-x_1)\Delta^2 f(x_0)$

$+ (x-x_0)(x-x_1)(x-x_2)\Delta^3 f(x_0) + ---$

for this construct divided difference table

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
300	2.4771			
		0.00145		
304	2.4829		$-0.00001 \cong 0$	
		0.0014		$0.0000014286 \cong 0$
305	2.4843		0	
		0.0014		
307	2.4871			

for $f(301)$, $x = 301$, $x_0 = 300$, $x_1 = 304$, $x_2 = 305$,
$x_3 = 307$

$f(301) = 2.4771 + (301-300)\times 0.00145 + (301-300)\times(301-304) \times(-0.00001)$

$+ (301-300)(301-304)(301-305)\times(0.0000014286)$

or $2.4771 + 1\times 0.00145 + 0 + 0 = 2.47855 \cong 2.4785$

$= 2.4771 + 0.00145 + 0.00003 + 0.0000171429$

$= 2.4785971429$

$\cong 2.4786$

Hence $f(301) = 2.4786$ (Approx).