Total differential Equation

Necassary and sufficient condition for integrability of the total diff.^{n} Eq.^{n} $Pdx+Qdy+Rdz=0$

$$\boxed{P\left(\frac{\partial Q}{\partial z}-\frac{\partial R}{\partial y}\right)+Q\left(\frac{\partial R}{\partial x}-\frac{\partial P}{\partial z}\right)+R\left(\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}\right)=0}$$

* The condition of Exactness:-

If the integral of Total differential Eq.^{n} $Pdx+Qdy+Rdz=0$ is $u(x,y,z)=c$

If this is Exact D.E.

$$Pdx+Qdy+Rdz=0=\frac{\partial u}{\partial x}.dx+\frac{\partial u}{\partial y}.dy+\frac{\partial u}{\partial z}.dz$$

$$\Rightarrow P=\frac{\partial u}{\partial x},\quad Q=\frac{\partial u}{\partial y},\quad R=\frac{\partial u}{\partial z}$$

from first two term

$$\frac{\partial P}{\partial y}=\frac{\partial^2 u}{\partial y\,\partial x},\quad \frac{\partial Q}{\partial x}=\frac{\partial^2 u}{\partial x\,\partial y}.$$

$$\Rightarrow \frac{\partial P}{\partial y}=\frac{\partial Q}{\partial x} \quad \text{—— (1)}$$

similarly from (i) & (iii) and (ii) & (iii) term

$$\frac{\partial P}{\partial z}=\frac{\partial R}{\partial x},\quad \frac{\partial Q}{\partial z}=\frac{\partial R}{\partial y} \quad \text{—— (2)}$$

so $P\left[\frac{\partial Q}{\partial z}-\frac{\partial R}{\partial y}\right]+Q\left(\frac{\partial R}{\partial x}-\frac{\partial P}{\partial z}\right)+R\left(\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}\right)=0$

from Eq^n (1) & (2) we have

$\Rightarrow P(0) + Q(0) + R(0) = 0$

$\Rightarrow 0 = 0$

Hence it is the required condition for exactness of $diff^n$ eq^n.

Method of inspection:-

(i) $x\,dy + y\,dx = d(xy)$

(ii) $xy\,dz + xz\,dy + yz\,dx = d(xyz)$

(iii) $\dfrac{x\,dy - y\,dx}{x^2} = d\left(\dfrac{y}{x}\right)$

(iv) $\dfrac{y\,dx - x\,dy}{x^2+y^2} = d\left(\tan^{-1}\dfrac{y}{x}\right)$

(v) $\dfrac{x\,dy + y\,dx}{xy} = d[\log(xy)]$

(vi) $\dfrac{x\,dx + y\,dy}{x^2+y^2} = d\left\{\frac{1}{2}\log(x^2+y^2)\right\}$

Ex:→ Solve. $\underbrace{(yz+xyz)}_{P}dx + \underbrace{(zx+xyz)}_{Q}dy + \underbrace{(xy+xyz)}_{R}dz = 0$

Sol^n, condition is

$$P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = 0$$

$$P\cdot(\cancel{x} + xy - \cancel{x} - xz) + Q(\cancel{y} + yz - \cancel{y} - xy) + R(\cancel{z} + xz - \cancel{z} - yz) = 0$$

$\Rightarrow$ The given TDE satisfied the condition of integrability.

Divide Entire Eq.^{n} by xyz.

$$\left(\frac{1}{x}+1\right)dx+\left(\frac{1}{y}+1\right)dy+\left(\frac{1}{z}+1\right)dz=0$$

On Integration

$$\log x + x + \log y + y + \log z + z = c$$

where c is the constant of integration

$$\boxed{\log(xyz)+(x+y+z)=c} \quad \text{Ans}$$

Ex:→ Solve. $\underbrace{(y^2+z^2-x^2)}_{P}dx\underbrace{-2xy}_{Q}dy\underbrace{-2xz}_{R}dz=0$ —(i)

Sol^n:→ $P\left(\frac{\partial Q}{\partial z}-\frac{\partial R}{\partial y}\right)+Q\left(\frac{\partial R}{\partial x}-\frac{\partial P}{\partial z}\right)+R\left(\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}\right)$

$\Rightarrow P(0-0)+Q(-2z-2z)+R(2y+2y)$

$=(-2xy)(-4z)+(-2xz)(4y)$

$=0$

Hence the given TDE satisfy the condition of Integrability.

$(y^2+z^2-x^2)dx = 2x(ydy+zdz)$

$\Rightarrow (y^2+z^2+x^2-2x^2)dx = 2x(ydy+zdz)$

$\Rightarrow (x^2+y^2+z^2)dx = 2x(xdx+ydy+zdz)$

$\Rightarrow \dfrac{dx}{x} = \dfrac{2(xdx+ydy+zdz)}{x^2+y^2+z^2}$

On Integration

$$\log x = \log(x^2+y^2+z^2) + \log c$$

$$\log x = \log(x^2+y^2+z^2)c$$

$$\boxed{x = c(x^2+y^2+z^2)}$$