2006 (12.ch)

Sensitivity Analysis

The optimal sol.ⁿ of LPP (max $z = cx$, s.t $Ax \leq b$, $x \geq 0$) dependents upon parameter (c_j, a_{ij}, b_i) of the problem, The parameter of the problem are usually not known with complete certainty i e a_{ij}, b_i, c_j are esitamate or they vary over time

Thus the investigation that deals with changes in the optimum solution due to changes in the parameter (a_{ij}, b_i, c_j) is called Sensitivity Analysis

Q. Min $Z = 6x_1 + 7x_2 + 3x_3 + 5x_4$

s.t $\quad 5x_1 + 6x_2 - 3x_3 + 4x_4 \geq 12$

$x_2 + 5x_3 - 6x_4 \geq 10$

$2x_1 + 5x_2 + x_3 + x_4 \geq 8$

& $x_1, x_2, x_3, x_4 \geq 0$

Sol.

Max $Z' = -6x_1 - 7x_2 - 3x_3 - 5x_4$

s.t $\quad -5x_1 - 6x_2 + 3x_3 - 4x_4 \leq -12$

$+0x_1 - x_2 - 5x_3 + 6x_4 \leq -10$

$-2x_1 - 5x_2 - x_3 - x_4 \leq -8$

& $x_1, x_2, x_3, x_4 \geq 0$

by adding slack variable x_5, x_6 & x_7

Max $Z' = -6x_1 - 7x_2 - 3x_3 - 5x_4 + 0x_5 + 0x_6 + 0x_7$

s.t $\quad -5x_1 - 6x_2 + 3x_3 - 4x_4 + x_5 = -12$

$0x_1 - x_2 - 5x_3 + 6x_4 + x_6 = -10$

$-2x_1 - 5x_2 - x_3 + x_4 + x_7 = -8$

construct a table

		C_j	-6	-7	-3	-5	0	0	0
	C_B	X_B	x_1	x_2	x_3	x_4	x_5	x_6	x_7
x_5	0	-12	-5	$\boxed{-6}$	3	-4	1	0	0
x_6	0	-10	0	-1	-5	6	0	1	0
x_7	0	-8	-2	-5	-1	-1	0	0	1
	$Z_j - C_j \rightarrow$		6	7 ↑	3	5	0	0	0

To find the leaving vector β_r

$$x_{B_r} = \min\left[x_{\beta_i} : x_{\beta_i} < 0\right]$$

$$= \min\left[x_{\beta_1}, x_{\beta_2}, x_{\beta_3}\right]$$

$$= \min\left[-12, -10, -8\right] = -12 = x_{\beta_1}$$

$\therefore r = 1$

To find the entering vector a_k when $r=1$

$$\frac{\Delta_k}{x_{rk}} = \max_j\left[\frac{\Delta_j}{x_{rj}}, x_{rj} < 0\right]$$

$$\frac{\Delta_k}{x_{1k}} = \max_j\left[\frac{\Delta_j}{x_{1j}}, x_{1j} < 0\right]$$

$$= \max_j\left[\frac{\Delta_1}{x_{11}}, \frac{\Delta_2}{x_{12}}, \frac{\Delta_4}{x_{14}}\right]$$

$$= \max_j\left[\frac{6}{-5}, \frac{7}{-6}, \frac{5}{-4}\right]$$

$$= -7/6 = \frac{\Delta_2}{x_{12}}$$

Hence $k=2$

So we must enter the vector $r=1, k=2$

The key element is -6

	c_j		-6	-7	-3	-5	0	0	0
	c_B	X_B	x_1	x_2	x_3	x_4	x_5	x_6	x_7
x_2	-7	2	$5/6$	1	$-1/2$	$2/3$	$-1/6$	0	0
x_6	0	-8	$5/6$	0	$\boxed{-11/2}$	$20/3$	$-1/6$	1	0
x_7	0	2	$13/6$	0	$-7/2$	$7/3$	$-5/6$	0	1
	$z_j - c_j \rightarrow$		$1/6$	0	$13/2$	$1/3$	$7/6$	0	0

Here only x_{B_2} is less than zero so x_{B_2} is leaving vector so $r = 2$

To find entering vector for $r = 2$

$$\frac{\Delta_K}{\Delta_{2K}} = \max_j \left[\frac{\Delta_j}{x_{2j}} , x_{2j} < 0 \right]$$

$$= \max_j \left[\frac{\Delta_3}{x_{23}} , \frac{\Delta_5}{x_{25}} \right]$$

$$= \max_j \left[\frac{13/2}{-11/2} , \frac{7/6}{-1/6} \right]$$

$$= \max_j \left[-\frac{13}{11} , -7 \right] = -\frac{13}{11} = \frac{\Delta_3}{x_{23}}$$

Hence, $K = 3$

So we must enter the vector $r = 2$, $K = 3$

The key element is $-11/2$

		c_j	-6	-7	-3	-5	0	0	0
	C_B	X_B	x_1	x_2	x_3	x_4	x_5	x_6	x_7
x_2	-7	$30/11$	$25/33$	1	0	$2/33$	$-5/33$	$-1/11$	0
x_3	-3	$16/11$	$-5/33$	0	1	$-40/33$	$1/33$	$-2/11$	0
x_7	0	$78/11$	$4/11$	0	0	$21/11$	$-8/11$	$-7/11$	1
	$z_j - c_j \rightarrow$		$38/33$	0	0	$271/33$	$32/33$	$13/11$	0

So the solution is $x_1 = 0$, $x_2 = \frac{30}{11}$, $x_3 = \frac{16}{11}$, $x_4 = 0$ is feasible

So required sol^n

$$\text{Min } z = 6 \times 0 + 7 \times \frac{30}{11} + 3 \times \frac{16}{11} + 5 \times 0$$

$$\text{Min } z = \frac{258}{11}$$

check $x_1 = \frac{16}{13}$, $x_2 = \frac{6}{13}$, $x_3 = \frac{8}{13}$, min $z = \frac{2280}{13}$

The change in LPP which are study by sensitivity analysis include
1 coeff. c_j of objective function
(a) coeff." of basic variable $c_j \in C_B$,
(b) coeff." of non basic variable $c_j \notin C_B$

2 Change in the R.H.S constant i.e b_i
3 change in a_{ij} i.e component of matrix A
this include
(a) coeff." of basic variable $a_{ij} \in B$
(b) coeff." of non basic variable $a_{ij} \notin B$

4. Addition of new variable to the problem
5. Addition of new (secondary) constraint

The change may result in one of the three following cases:
case 1:
optimal solution remain unchanged i.e the basic variable and their values remains unchanged

case 2: Basic variable remains the same but their values are changed

case 3: The basic solution changes completely lemma: if $d_j \geq 0$, $d_j + \lambda d_j' \geq 0$, $j = 1, 2, \ldots n$ then

$$\max_{\substack{j \\ (d_j' > 0)}} \left[\frac{-d_j}{d_j'}\right] \leq \lambda \leq \min_{\substack{j \\ (d_j' < 0)}} \left[\frac{-d_j}{d_j'}\right]$$

proof: since $d_j \geq 0$ & $d_j + \lambda d_j' \geq 0$ ($j = 1, 2, \ldots n$)
we have three possibilities

case 1:

for those j for which $d_j' > 0$ we have

$$\lambda \geq \frac{-d_j}{d_j'} \quad \text{or} \quad \lambda \geq \max_j \left[\frac{-d_j}{d_j'}\right] \quad (d_j' > 0)$$

case 2:

for those j for which $d_j' < 0$, we have

$$\lambda \leq \frac{-d_j}{d_j'} \quad \text{or} \quad \lambda \leq \min_j \left[\frac{-d_j}{d_j'}\right] \quad (d_j' < 0)$$

$(d_j + \lambda d_j' \geq 0)$

case 3:

for those value for which $d_j' = 0$, we have λ as unrestricted

Thus from 1 and 2 we conclude that λ must satisfies the relationship

$$\max_{\substack{j \\ (d_j' > 0)}} \left[\frac{-d_j}{d_j'}\right] \leq \lambda \leq \min_{\substack{j \\ (d_j' < 0)}} \left[\frac{-d_j}{d_j'}\right]$$

changes in the coefficient c_j is the objective function

Let us consider the L.P.P

$$\left.\begin{array}{ll} & \text{Max. } Z = CX \\ \text{s.t} & AX \leq b \\ & \& \ X \geq 0 \end{array}\right] - (1)$$

for which X_B is the optimal basic feasible solution and B is the optimal basic matrix

we assume that our problem is non degenerate

So we have a basic feasible sol^n to (1)

$$X_B = B^{-1} b$$

if we change some component c_j of vector c then X_B will not change (as independent of c) but the condition

$\Delta_j = Z_j - c_j \geq 0$ for all j, which are satisfies for the optimum sol^n X_B before changing c_j will not be satisfied.

when c_j is change to $\hat{c}_j$, the variation in vector c can be ~~[illegible]~~ made in two possible ways.

1) Only that component c_j of vector c is change which is the coefficient of non basic variable x_j in the objective function i.e $c_j \notin c_B$

2) Only that component c_j of vector c is change which is the coefficient of basic variable x_j in the objective function i.e $c_j \in c_B$

Case 1 : $c_j \notin c_B$

Let us assume that $c_j \notin c_B$ is change to $c_j + \Delta c_j$

$\because$ since X_B is the optimal soln

$\therefore z_j - c_j \geq 0 \quad \forall j$

if with this change the optimality of X_B is to be preserved then Δc_j must satisfied the condition

$$\Delta_j' \geq 0$$

$$\Delta_j = z_j - c_j \geq 0$$

$$\Rightarrow z_j - (c_j + \Delta c_j) \geq 0$$

$$\Rightarrow z_j - c_j - \Delta c_j \geq 0$$

$$\Rightarrow \Delta_j \geq \Delta c_j$$

$$\Rightarrow \Delta c_j \leq \Delta_j$$

Thus

~~doas~~ if $c_j \notin c_B$ is replaced by $c_j + \Delta c_j$ s.t $\Delta c_j \leq \Delta_j$

Then the optimum value of the objective fun" and the optimal soln will not be effected.

case 2 : variation in $c_j \in c_B$ (say c_{Br})

$$z_j = c_B X_j = \sum_{i=1}^{m} c_{Bi} x_{ij}$$

if we change c_{Br} to $c_{Br} + \Delta c_{Br}$ then the value of z_j is effected and is given by

$$\hat{z}_j = \left[\sum_{i=1, i\neq r}^{m} c_{Bi} x_{ij} + (c_{Br} + \Delta c_{Br}) x_{rj} \right]$$

also with this change

$$\hat{z}_j - c_j = \left[\sum_{i=1, i\neq r}^{m} c_{Bi} x_{ij} + (c_{Br} + \Delta c_{Br}) x_{rj} \right] - c_j$$

$$\hat{z}_j - c_j = \left[\sum_{i=1}^{m} c_{Bi} x_{ij} + \Delta c_{Br} x_{rj} \right] - c_j$$

$$= (z_j - c_j) + \Delta c_{Br} x_{rj} \quad — (*)$$

if the solution remains optimal after such variation then Δc_{Br} must satisfy the optimulity condition,

$$\left.\begin{array}{l} \hat{z}_j - c_j \geq 0 \\ \text{or} \quad z_j - c_j + x_{rj} \Delta c_{Br} \geq 0 \ (\text{by } (*)) \end{array}\right] — (1)$$

since $z_j - c_j \geq 0$ (as x_B is optimal initial

So by using lemma initially replacing

$d_j \rightarrow z_j - c_j$, $d_j' \rightarrow x_{rj}$ & $\lambda \rightarrow \Delta c_{Br}$

$$\max_{\substack{j \\ (x_{rj} > 0)}} \left[\frac{-\Delta_j}{x_{rj}} \right] \leq \Delta c_{Br} \leq \min_{\substack{j \\ (x_{rj} < 0)}} \left[\frac{-\Delta_j}{x_{rj}} \right] \quad — (**)$$

and the value of $\hat{z} = z + \Delta c_{Br} x_{Br}$ — $(***)$