Eqⁿ of Tangent (स्पर्श रेखा) –

At the pt. (x_1, y_1)

$$y - y_1 = \left(\frac{dy}{dx}\right)_{(x_1,y_1)} (x - x_1) \quad \text{—(1)}$$

[$\frac{dy}{dx}$: slope at (x_1, y_1) of ellipse]

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

diff. wrt $x \Rightarrow \frac{2x}{a^2} + \frac{2y}{b^2}\frac{dy}{dx} = 0$

$$y\frac{dy}{dx} = \frac{-x}{a^2}$$

$$\text{or } \frac{dy}{dx} = \frac{-b^2}{a^2}\frac{x}{y}$$

$$\therefore \left(\frac{dy}{dx}\right)_{(x_1,y_1)} = \frac{-b^2}{a^2}\frac{x_1}{y_1} \quad \text{—(2)}$$

Put in (1), $y - y_1 = \left(\frac{-b^2}{a^2}\right)\left(\frac{x_1}{y_1}\right)(x - x_1)$

$$y - y_1 = \frac{-b^2 x x_1}{a^2 y_1} + \frac{b^2 x_1^2}{a^2 y_1}$$

$$\boxed{\cancel{\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1}}$$

PTO

EW 9.1

$$y-y_1=\left(\frac{-b^2}{a^2}\right)\left(\frac{x_1}{y_1}\right)(x-x_1)$$

$$y-y_1=\frac{-b^2xx_1}{a^2y_1}+\frac{b^2}{a^2}\frac{x_1^2}{y_1}$$

$$y-y_1=\frac{-b^2xx_1+b^2x_1^2}{a^2y_1}$$

$$yy_1a^2-a^2y_1^2=-b^2xx_1+b^2x_1^2$$

Divide by a^2b^2

$$\frac{yy_1a^2}{a^2b^2}-\frac{a^2y_1^2}{a^2b^2}=\frac{-b^2xx_1}{a^2b^2}+\frac{b^2x_1^2}{a^2b^2}$$

$$\frac{yy_1}{b^2}-\frac{y_1^2}{b^2}=\frac{-xx_1}{a^2}+\frac{x_1^2}{a^2}$$

or $$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}=1$$

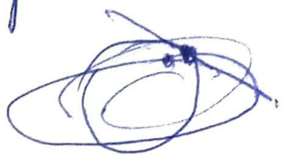
($\because (x_1,y_1)$ lies on Ellipse)

or $$\boxed{\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1}$$

उदाहरण, P16

Ellipse (e)

Ex.4. A circle of radius r is concentric (i.e. ~~same~~ common centre) with the ellipse $x^2/a^2+y^2/b^2=1$. Prove that the common tangent is inclined to the major axis at an angle $\tan^{-1}\left[\sqrt{\frac{r^2-b^2}{a^2-r^2}}\right]$ where a & b are the semi-axes of the ellipse. ~~Find~~



~~also length also~~

Sol Ellipse: $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ —(1)

Concentric circle $x^2+y^2=r^2$ —(2)

Tangent ([illegible]) of ellipse in slope form ([illegible]) (slope form is taken ∵ of presence of $\tan^{-1}$ $m=\tan\phi$)

$$y=mx+\sqrt{a^2m^2+b^2}$$

$$\text{or } mx-y+\sqrt{a^2m^2+b^2}=0 \quad —(3)$$

Since line (3) touches circle (2)

∴ the length of perpendicular (or [illegible]) from centre of circle (0,0) = radius of circle (property of tangent)

Ellipse

$\therefore$ Length of perp $= r = \dfrac{m\cdot 0 - 1\cdot 0 + \sqrt{a^2m^2+b^2}}{\sqrt{m^2+1^2}}$

$r\sqrt{m^2+1} = \sqrt{a^2m^2+b^2}$

Square both sides

$r^2(m^2+1) = a^2m^2+b^2$ or $r^2m^2+r^2 = a^2m^2+b^2$

or $m^2(a^2-r^2) = r^2-b^2$

$m^2 = \dfrac{r^2-b^2}{a^2-r^2} \Rightarrow m = \sqrt{\dfrac{r^2-b^2}{a^2-r^2}}$

Slope of (CT) $= m = \tan\phi = \sqrt{\dfrac{r^2-b^2}{a^2-r^2}}$

$\therefore \phi = \tan^{-1}\left[\sqrt{\dfrac{r^2-b^2}{a^2-r^2}}\right]$

where ϕ is angle between common tangent & major axis.

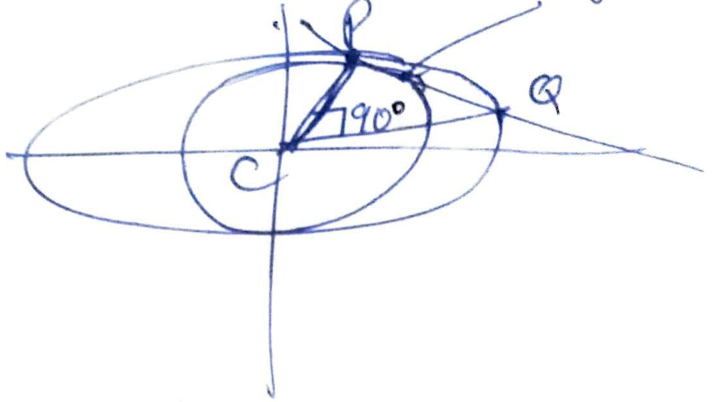
<u>EX3</u> If the portion of the line $x\cos\alpha + y\sin\alpha = p$ intercepted ([illegible]) subtend a right-angle (90°) at centre of ellipse prove that the <u>line touches the circle</u> of radius $ab/\sqrt{a^2+b^2}$ concentric with the ellipse.

Sol

PQ touches circle.

ellipse

9



ellipse $x^2/a^2 + y^2/b^2 = 1$ —(1)

line $x\cos\alpha + y\sin\alpha = p$ —(2)

To get joint eqn (संयुक्त समीकरण) of CP & CQ & make them homogeneous (समघातीय) powers of variables $= 1^2$

homogeneous ⇒ all variables have same power

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \left(\frac{x\cos\alpha + y\sin\alpha}{p}\right)^2$$

ex. $2xy$, x^2, $1+1=2$ in $2xy\cos\alpha\sin\alpha$, 2 in y^2

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{x^2\cos^2\alpha + 2xy\cos\alpha\sin\alpha + y^2\sin^2\alpha}{p^2}$$

$$\frac{p^2 x^2}{a^2} + \frac{p^2 y^2}{b^2} - x^2\cos^2\alpha - y^2\sin^2\alpha - 2xy\cos\alpha\sin\alpha = 0$$

Or $$\frac{x^2}{a^2} - \frac{x^2\cos^2\alpha}{p^2} - \frac{2xy\cos\alpha\sin\alpha}{p^2} + \frac{y^2}{b^2} - \frac{y^2\sin^2\alpha}{p^2} = 0$$

$$x^2\left[\frac{1}{a^2} - \frac{\cos^2\alpha}{p^2}\right] - 2\frac{\sin\alpha\cos\alpha}{p^2}xy + y^2\left[\frac{1}{b^2} - \frac{\sin^2\alpha}{p^2}\right] = 0$$ —(3)

In (3), CP will be at $90°$ to CQ if ellipse (h)

Coefficient of x^2 + coeff of $y^2 = 0$
(गुणांक)

$$\text{or } \left(\frac{1}{a^2} - \frac{\cos^2\alpha}{p^2}\right) + \left(\frac{1}{b^2} - \frac{\sin^2\alpha}{b^2}\right) = 0$$

$$\text{or } \frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{p^2}\left(\cos^2\alpha + \sin^2\alpha\right)$$

$$\text{or } \frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{p^2} \quad \text{———} (4)$$

same HP

Eqn of concentric circle with centre (0,0) & radius $\frac{ab}{\sqrt{a^2+b^2}}$ (given in question) is

$$x^2 + y^2 = \frac{a^2b^2}{a^2+b^2} \quad \text{———} (5)$$

if line (2) touches circle (5) then length of perp from centre (0,0) = radius of circle

$$\frac{0\cdot\cos\alpha + 0\cdot\sin\alpha - p}{\sqrt{\cos^2\alpha + \sin^2\alpha}} = \sqrt{\frac{a^2b^2}{a^2+b^2}}$$

~~0 + 0 + 0~~

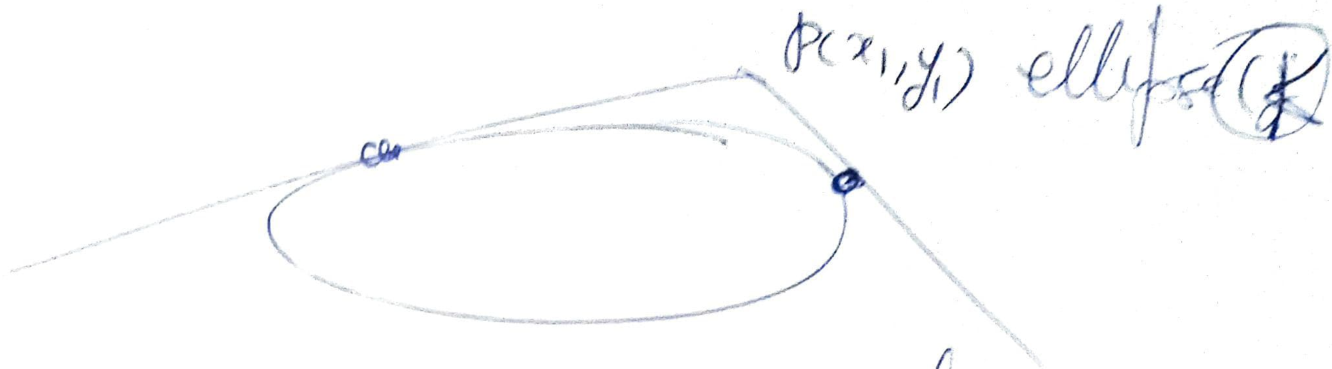
Squaring both sides

$$\text{or } \frac{p^2}{1} = \frac{a^2b^2}{a^2+b^2}$$

~~divide by a^2b^2~~

$$\text{or } \frac{1}{p^2} = \frac{a^2+b^2}{a^2b^2} = \frac{1}{b^2} + \frac{1}{a^2} = (4)$$

(सिद्ध किया) Hence Proved



Eqⁿ of Pair of Tangents from an external pt to ellipse $x^2/a^2+y^2/b^2=1$

(स्पर्शी युग्म का समीकरण)

$$SS_1=T^2$$

$$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)\left(\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1\right)=\left(\frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1\right)^2$$

Director Circle (नियामक वृत्त) —

The locus (बिंदुपथ) of the point of intersection of 2 perpendicular tangents of an ellipse is called the director circle.

(दीर्घवृत्त की 2 लंबवत् रेखाओं के प्रतिच्छेद बिंदु के बिंदुपथ को नियामक वृत्त कहते हैं)

Eqⁿ of Director Circle : ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

~~Eqⁿ of~~ Pt. of intersection of 2 perp tangents is (x_1,y_1)

गोरखपुर P210

Ex. 8 Prove that the line $lx+my+n=0$ is a normal to the ellipse $x^2/a^2+y^2/b^2=1$ if

$$\frac{a^2}{l^2}+\frac{b^2}{m^2}=\frac{(a^2-b^2)^2}{n^2}$$

Thats why we take parametric eqn

Sol Let $P(a\cos\theta, b\sin\theta)$ be any pt. on the ellipse.

eqn of Normal (आवश्यक) $ax\sec\theta-by\csc\theta=a^2-b^2$ —(1)

& line: $lx+my+n=0$ —(2)

if line (2) is Normal to ellipse then it is same as line (1)

∴ Comparing Coefficients (गुणांकों की तुलना)

$$\frac{a\sec\theta}{l}=\frac{-b\csc\theta}{m}=\frac{-(a^2-b^2)}{n}$$

~~Or $a\sec\theta$~~

~~$\therefore \cos\theta=\frac{-ln}{l(a^2-b^2)}$~~

$$\frac{\sec\theta}{l}=\frac{-(a^2-b^2)}{na}$$

$\left(\sec\theta=\frac{1}{\cos\theta}\right)$

$$\cos\theta\cdot l=-\frac{na}{(a^2-b^2)}$$

$$\therefore \cos\theta=\frac{-na}{l(a^2-b^2)} \quad —(3)$$

$$\frac{-b\csc\theta}{m}=-\frac{(a^2-b^2)}{n}$$

$$\frac{-m}{b\csc\theta}=-\frac{n}{a^2-b^2}$$

$$\cancel{+}\frac{m}{b}\sin\theta=\cancel{+}\frac{n}{a^2-b^2}$$

$$\sin\theta=\frac{bn}{m(a^2-b^2)} \quad —(4)$$

Sq. & add (3) & (4) ellipse (9)

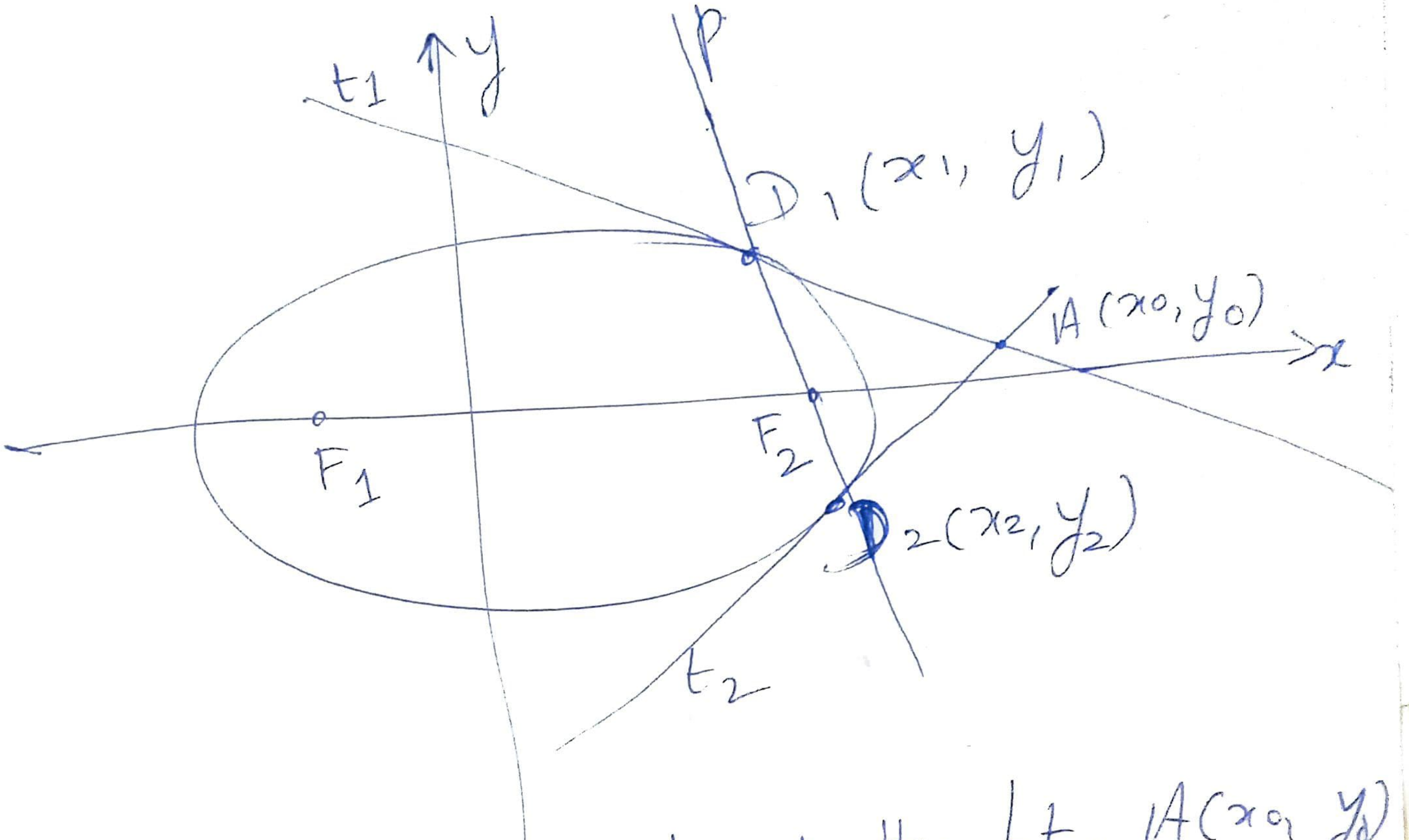
$$\cos^2\theta + \sin^2\theta = \frac{a^2n^2}{l^2(a^2-b^2)^2} + \frac{b^2n^2}{m^2(a^2-b^2)^2}$$

or $$1 = \frac{a^2n^2}{l^2(a^2-b^2)^2} + \frac{b^2n^2}{m^2(a^2-b^2)^2}$$

$$\frac{a^2}{l^2} + \frac{b^2}{m^2} = \frac{(a^2-b^2)^2}{n^2}$$ — required condition.

(Internet)

Diagram of Pole & Polar in Ellipse



Eqn of polar p of the pt. $A(x_0, y_0)$

Defn: If from a pt. $A(x_0, y_0)$, exterior to the ellipse, drawn are tangents, then the secant line passing thru the contact pts $D_1(x_1, y_1)$ and $D_2(x_2, y_2)$ is the **Polar** of the pt. A. The pt. A is called the **Pole**.

Eqn of Polar: $\frac{xx_0}{a^2} + \frac{yy_0}{b^2} = 1$

PTO

Coords of Pole -

Pole of line $lx+my+n=0$ wrt ellipse

$x^2/a^2 + y^2/b^2 = 1$;

$lx+my+n=0$ — (1)

Let (x_1, y_1) be Pole of

eqn of Polar : $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$ — (2)

But (1) is Polar of (x_1, y_1) $\therefore$ (1) & (2)

represent the same line

∴ Comparing Coeffs

$$\frac{x_1/a^2}{l} = \frac{y_1/b^2}{m} = \frac{-1}{n}$$

$$\therefore x_1 = \frac{-a^2 l}{n}, \quad y_1 = \frac{-b^2 m}{n}$$

$$\boxed{\therefore \text{Pole}: \left(\frac{-a^2 l}{n}, \frac{-b^2 m}{n}\right)}$$

(संयुग्मी ~~बिंदु~~ ~~line~~ रेखाएँ / 2 lines)

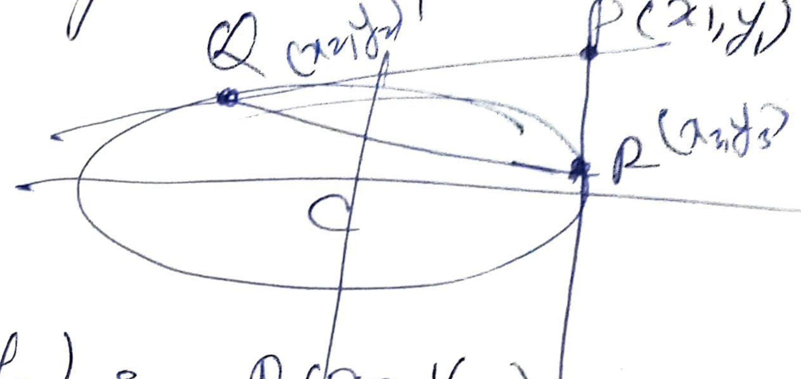
Conjugate ~~planes~~ lines :- If pole of one line lies on the other, the lines are called conjugate or reciprocal lines wrt given conic.

(2 रेखाएं एक दीर्घवृत्त के सापेक्ष संयुग्मी रेखाएं कहलाती हैं जबकि एक रेखा का ध्रुव अन्य रेखा पर विद्यमान हो)

Conjugate Points (संयुग्मी बिंदु) - 2 pts P & Q which are s.t. if each lies on the polar of the other wrt to a conic are conjugate pts wrt conic.

Chord of Contact of Tangents of ellipse (स्पर्शरेखाओं की स्पर्श जीवा)

The line joining the pts of contact of tangents drawn from an external pt. P to the ellipse is called the chord of contact of tangents to the ellipse wrt P.



Pts of contact of tangents are $Q(x_2,y_2)$ & $R(x_3,y_3)$

QR is the chord of contact of tangents.

eqn of tangent at Q & R are:

$$\frac{xx_2}{a^2}+\frac{yy_2}{b^2}=1 \quad \text{---(1)}$$

$$\frac{xx_3}{a^2}+\frac{yy_3}{b^2}=1 \quad \text{---(2)}$$

Both (2) & (3) pass thru $P(x_1,y_1)$:

$$\frac{x_1x_2}{a^2}+\frac{y_1y_2}{b^2}=1 \quad \text{---(3)}$$

$$\frac{x_1x_3}{a^2}+\frac{y_1y_3}{b^2}=1 \quad \text{---(4)}$$

of Q

(3) & (4) $\Rightarrow$ Pts (x_2, y_2) & (x_3, y_3)

lie on line $xx_1/a^2 + yy_1/b^2 = 1$ ($\because x_1$ is common)

$\therefore$ locus of (x_2, y_2) & (x_3, y_3) i.e., eqn of QR is $\boxed{\dfrac{xx_1}{a^2} + \dfrac{yy_1}{b^2} = 1}$

?

प्रश्नावली P38
Ex. 7

Show that the locus of the poles of normal chords of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is the curve $\frac{a^6}{x^2}+\frac{b^6}{y^2}=(a^2-b^2)^2$

(सिद्ध करो कि ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ की अभिलंब जीवाओं के ध्रुवों का बिंदुपथ है)

Sol. eqn of Normal (अभिलंब) to ellipse is $ax\sec\theta - by\operatorname{cosec}\theta = a^2-b^2$ —(1)

Let Pole be ध्रुव (x_1,y_1) of this Normal

then (1) would be ~~be~~ Polar of (x_1,y_1) wrt ellipse (by def of Pole & Polar)

But given Polar is $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$ —(2)

(1) & (2) are same i.e., Polar

∴ Comparing coeff's (गुणांकों की तुलना) (2)

$$\frac{x_1/a^2}{a\sec\theta} = \frac{y_1/b^2}{-b\,\mathrm{cosec}\,\theta} = \frac{1}{a^2-b^2}$$

$$\frac{x_1\cos\theta}{a^3} = \frac{-y_1\sin\theta}{b^3} = \frac{1}{a^2-b^2}$$

$$\cos\theta = \frac{a^3}{x_1(a^2-b^2)}, \quad \sin\theta = \frac{-b^3}{y_1(a^2-b^2)}$$

Square & adding

$$1 = \frac{a^6}{x_1^2(a^2-b^2)^2} + \frac{b^6}{y_1^2(a^2-b^2)^2}$$

Multiply thru-out by $(a^2-b^2)^2$

$$\frac{a^6}{x_1^2} + \frac{b^6}{y_1^2} = (a^2-b^2)^2$$

∴ locus (बिंदुपथ) of (x_1, y_1) is

$$\frac{a^6}{x^2} + \frac{b^6}{y^2} = (a^2-b^2)^2$$ Hence

Proved

सिद्ध हुआ

गोरखपुर P39

Q. 8 Prove that the locus of the poles wrt the ellipse $x^2/a^2+y^2/b^2=1$ of any tangent to the auxiliary circle is the curve:

$$x^2/a^4+y^2/b^4=1/a^2$$

Ellipse ⑧

Pf. ~~Direc~~ Auxiliary Circle (सहायक वृत्त) of ellipse $x^2/a^2+y^2/b^2=1$ is

$$x^2+y^2=a^2 \quad \text{——— (1)}$$

Let Pole (ध्रुव) be (x_1, y_1) wrt ellipse of tangent (स्पर्श रेखा) of the Auxiliary Circle ~~wrt ellipse~~ (दीर्घवृत्त के सापेक्ष, (1) की किसी स्पर्श रेखा का ध्रुव (x_1, y_1) है)

Then Polar of ~~(x_1~~ Pole (x_1, y_1) wrt ellipse (दीर्घवृत्त के सापेक्ष (x_1, y_1) की ध्रुवी का समीकरण) is:

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1 \quad \text{——— (2)}$$

ellipse(t)

But (2) is Tangent of Auxiliary Circle(1)

$\therefore$ Length of perpendicular (d or) from centre $(0,0)$ of circle ~~Radius~~ to (2) = Radius of circle

$$\therefore \frac{0 \cdot 1 + 0 \cdot 1 - 1}{\sqrt{x_1^2/a^4 + y_1^2/b^4}} = a$$

$$\text{or} \quad \frac{1}{a} = \sqrt{x_1^2/a^4 + y_1^2/b^4}$$

Squaring $\dfrac{1}{a^2} = \dfrac{x_1^2}{a^4} + \dfrac{y_1^2}{b^4}$

$\therefore$ locus (पथ) of (x_1, y_1) is

$$\frac{x^2}{a^4} + \frac{y^2}{b^4} = a^2 \quad \text{Hence Proved.}$$

P35