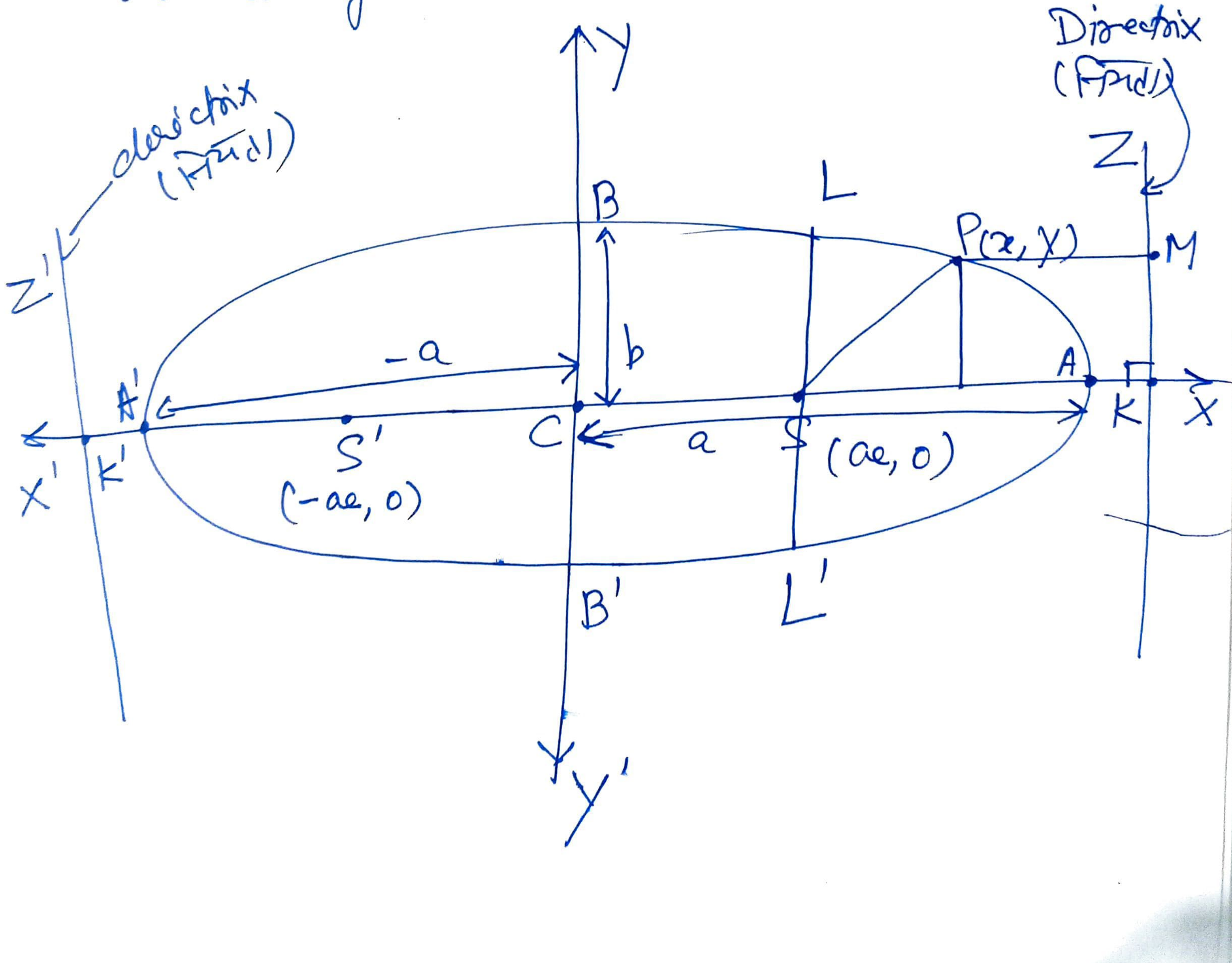
Ellipse (दीर्घवृत्त) - It is the locus (बिंदुपथ) of a point which moves such that (s.t.) its distance from a fixed point (स्थिर बिंदु) called <u>focus</u> (नाभि) to its distance from a <u>fixed straight line</u> called <u>directrix (नियता)</u> bears a constant ratio called <u>eccentricity (उत्केन्द्रता)</u> which is always less than 1. The eccentricity is denoted by e.



S: focus (नाभिक), ZK: directrix (नियता),
P(x, y): any point on ellipse, $e<1$

$$e = \frac{SP}{PM} \quad \text{or} \quad SP = e\,PM$$

Standard eqⁿ of Ellipse: $\boxed{\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1,\quad a>b}$

where $b^2 = a^2(1-e^2)$

Focii : $(ae, 0)$, $(-ae, 0)$

Directrix: $x = \frac{a}{e}$, $x = -\frac{a}{e}$

Major Axis (दीर्घ अक्ष) - AA' is called Major Axis.

Its length is $2a$.

Minor Axis (लघु अक्ष) - BB' is called Minor Axis.

Its length is $2b$.

When $b=a$, the ellipse becomes a Circle
$x^2+y^2=a^2$, radius $= a$.

General Eqⁿ of Ellipse: $e<1$, focus: (α, β),
directrix: $lx+my+n=0$

$$\boxed{(x-\alpha)^2+(y-\beta)^2 = \frac{e^2(lx+my+n)^2}{l^2+m^2}}$$

LL' is called the Latus Rectum (নাভিলম্ব) Ell of Ellipse. There are 2, Latus Rectums in an Ellipse.

Ex. Find Length of Latus Rectum.

Sol. $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, \quad a>b$

Coordinates (স্থানাংকগুলি) of L are (ae, SL)

Since L lies on Ellipse, $\frac{(ae)^2}{a^2}+\frac{(SL)^2}{b^2}=1,$

$$\frac{a^2e^2}{a^2}+\frac{SL^2}{b^2}=1$$

or $\frac{SL^2}{b^2}=1-e^2 \qquad$ or $SL^2=b^2(1-e^2)$

$\therefore SL=b\sqrt{1-e^2}$

But $b^2=a^2(1-e^2) \quad \therefore 1-e^2=\frac{b^2}{a^2}$ or $\sqrt{1-e^2}=\frac{b}{a}$

$\therefore SL=b\sqrt{1-e^2}=b\cdot\frac{b}{a}=\frac{b^2}{a}$

$\therefore$ Length of Latus Rectum $=\frac{2b^2}{a}$

$SL=b\sqrt{1-e^2}=b\cdot\frac{b}{a}=\frac{b^2}{a}$

Coordinates of $L:\left(ae,\frac{b^2}{a}\right)$, $L':\left(ae,-\frac{b^2}{a}\right)$

Similarly, for the other Latus Rectum,

$\left(-ae,\frac{b^2}{a}\right), \left(-ae,-\frac{b^2}{a}\right)$

Intercepts (अन्तःखण्ड)

when $y = 0$ then $x = \pm a$

Ellipse cuts the axis at $(a, 0)$ & $(-a, 0)$

when $x = 0$ then $y = \pm b$

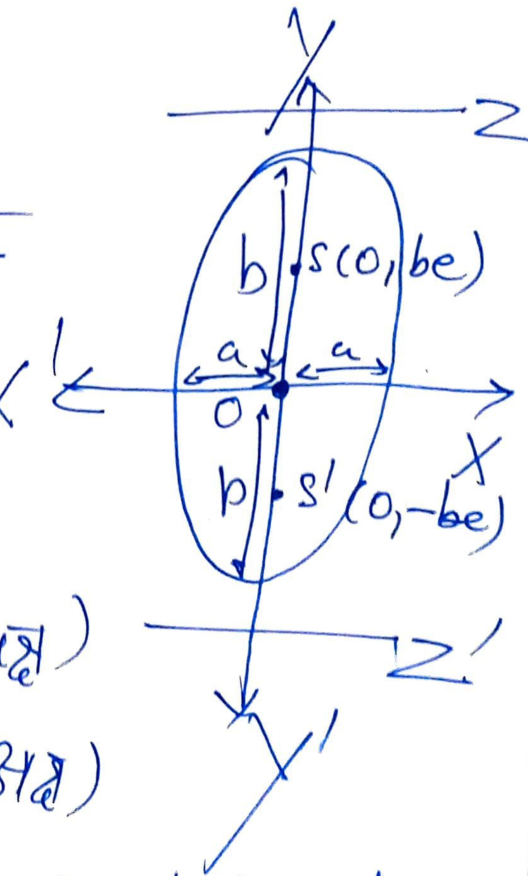
$\therefore$ y-axis is cut by the Ellipse at $(0, b)$ & $(0, -b)$

Region (क्षेत्र) -

Ellipse ~~lies to~~ lies between $x = \pm a$

and $y = \pm b$.

Second Form of the Ellipse -



$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

If $b > a$ then $a^2 = b^2(1 - e^2)$

x-axis is Minor Axis (लघु अक्ष)

y-axis is Major Axis (दीर्घ अक्ष)

Focii: S & S' are $(0, be)$, $(0, -be)$ respectively

Directrix: $y = \frac{b}{e}$, $y = -\frac{b}{e}$.

Coordinates of Latus Rectum: $\left(\frac{a^2}{b}, be\right)$, $\left(-\frac{a^2}{b}, be\right)$

Coordinates of other Latus Rectum: $\left(\frac{a^2}{b}, -be\right)$, $\left(-\frac{a^2}{b}, -be\right)$

Ell (5)

Ex. Find the Latus Rectum, e, foci & length of axes of the following ellipse.

$$3x^2+4y^2=12$$

Sol. Divide by 12.

$\frac{x^2}{4}+\frac{y^2}{3}=1$, Here $a>b$ $\therefore$ Ellipse is of Standard Form

$\therefore a=\sqrt{4}=2,\ b=\sqrt{3}$

$\therefore$ Length of Latus Rectum $=\frac{2b^2}{a}=\frac{2\times3}{2}=3$

$b^2=a^2(1-e^2)$ $\quad (\because a>b)$

$a^2(1-e^2)=b^2$ or $1-e^2=\frac{b^2}{a^2}$

$e^2=1-\frac{b^2}{a^2}=1-\frac{3}{4}=\frac{1}{4}$ $\quad \therefore e=1/2$

Foci : $(ae,0),\ (-ae,0)$

i.e., $(1,0),\ (-1,0)$

Major Axis $=2a=4$, Minor Axis $=2\sqrt{3}$

Ex. Find eqⁿ of Ellipse when focus is $(0,1)$, directrix is $x+y=0$, $e=1/2$. Ell (6)

Sol. Let P be any point (h,k) on Ellipse,

S is $(0,1)$,

By definition of e, $SP = e\,PM$ or $(SP)^2 = e^2(PM)^2$

Distance (...) $SP^2 = (h-0)^2 + (k-1)^2$ —(1)

$$PM^2 = \left(\frac{h\cdot 1 + k\cdot 1 + 0}{\sqrt{1^2+1^2}}\right)^2$$

[formula for perpendicular distance $(PM)^2$ from $P(h,k)$ on $x+y=0$]

Put in (1) $\Rightarrow$

$$h^2+k^2-2k+1 = \left(\frac{1}{2}\right)^2\left(\frac{h+k}{2}\right)^2$$

or $h^2+k^2-2k+1 = \frac{1}{4}\left(\frac{h+k}{2}\right)^2$

or $h^2+k^2-2k+1 = \frac{h^2+2hk+k^2}{8}$

or $8h^2+8k^2-16k+8-h^2-k^2-2hk=0$

or $7h^2+7k^2-16k+8-2hk=0$

$\therefore$ Locus of P is $7x^2+7y^2-16y+8-2xy=0$

which is eqⁿ of Ellipse.

Ex. Find the Latus Rectum, eccentricity, E-11 ⑦
foci & length of axes of the ellipse
$$9x^2+5y^2-30y=0.$$

Sol. We reduce this eqn to the standard form
$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, we need separate terms for
x^2 & y^2.

$$9x^2+5(y^2-6y)=0$$

$$9x^2+5\left[y^2-6y+\left(\tfrac{6}{2}\right)^2-\left(\tfrac{6}{2}\right)^2\right]=0$$ [Completing the square]

$$9x^2+5\left[y^2-6y+9-9\right]=0$$

$$9x^2+5(y^2-6y+9)-45=0$$

$$9x^2+5(y-3)^2=45$$

or $$\frac{x^2}{5}+\frac{(y-3)^2}{9}=1$$

or $$\frac{(x-0)^2}{5}+\frac{(y-3)^2}{9}=1$$

Shift the origin to $(0,3)$ [$\because$ of 0 in $x-0$ & 3 in $y-3$]

we get $\frac{x^2}{5}+\frac{y^2}{9}=1$

$a^2=5$, $b^2=9$, $b>a$

$\therefore a^2 = b^2(1-e^2)$

$5 = 9(1-e^2)$ or $1-e^2 = \frac{5}{9}$

or $e^2 = 1 - \frac{5}{9} = \frac{4}{9}$ $\therefore e = 2/3$

Length of Latus Rectum $= \frac{2a^2}{b} = \frac{2\times 5}{3} = \frac{10}{3}$

Focii $(0, \pm be)$

$be = 3\times\frac{2}{3} = 2$

$\therefore$ focii $(0,2), (0,-2)$

Due to shift in origin,

$(0+0, \overset{2+3}{\cancel{3+2}}), (0+0, \overset{-2+3}{\cancel{3+-2}})$

$\therefore$ Focii : $(0,5), (0,1)$.

Length of Major Axis $= 2b = 2\times 3 = 6$

" " Minor Axis $= 2a = 2\sqrt{5}$

Eq^n of Tangent (स्पर्श रेखा) –

At the pt. (x_1, y_1)

$$y - y_1 = \left(\frac{dy}{dx}\right)_{(x_1,y_1)} (x - x_1) \quad \text{—(1)}$$

$\left[\frac{dy}{dx}\right.$: slope at (x_1, y_1) of ellipse$\left.\right]$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

diff. wrt $x \Rightarrow \frac{\cancel{2}x}{a^2} + \frac{\cancel{2}y}{b^2}\frac{dy}{dx} = 0$

$$y\frac{dy}{dx} = \frac{-x}{a^2}$$

$$\text{or } \frac{dy}{dx} = \frac{-b^2}{a^2}\frac{x}{y}$$

$$\therefore \left(\frac{dy}{dx}\right)_{(x_1,y_1)} = \frac{-b^2}{a^2}\frac{x_1}{y_1} \quad \text{—(2)}$$

Put in (1), $y - y_1 = \left(\frac{-b^2}{a^2}\right)\left(\frac{x_1}{y_1}\right)(x - x_1)$

$$y - y_1 = \frac{-b^2 x x_1}{a^2 y_1} + \frac{b^2 x_1^2}{a^2 y_1}$$

$$\boxed{\frac{x x_1}{a^2} + \frac{y y_1}{b^2} = 1}$$

Eqn of Tangent in Slope Form

$$\boxed{y = mx \pm \sqrt{a^2m^2+b^2}}$$

Coordinates of point of Contact:

$$\left(\mp \frac{a^2m}{\sqrt{a^2m^2+b^2}}, \pm \frac{b^2}{\sqrt{a^2m^2+b^2}} \right)$$

Eqn of Normal (Perpendicular) –

At the pt. (x_1, y_1)

1st Form – $$\boxed{\frac{x-x_1}{\left(\frac{x_1}{a^2}\right)} = \frac{y-y_1}{\left(\frac{y_1}{b^2}\right)}}$$

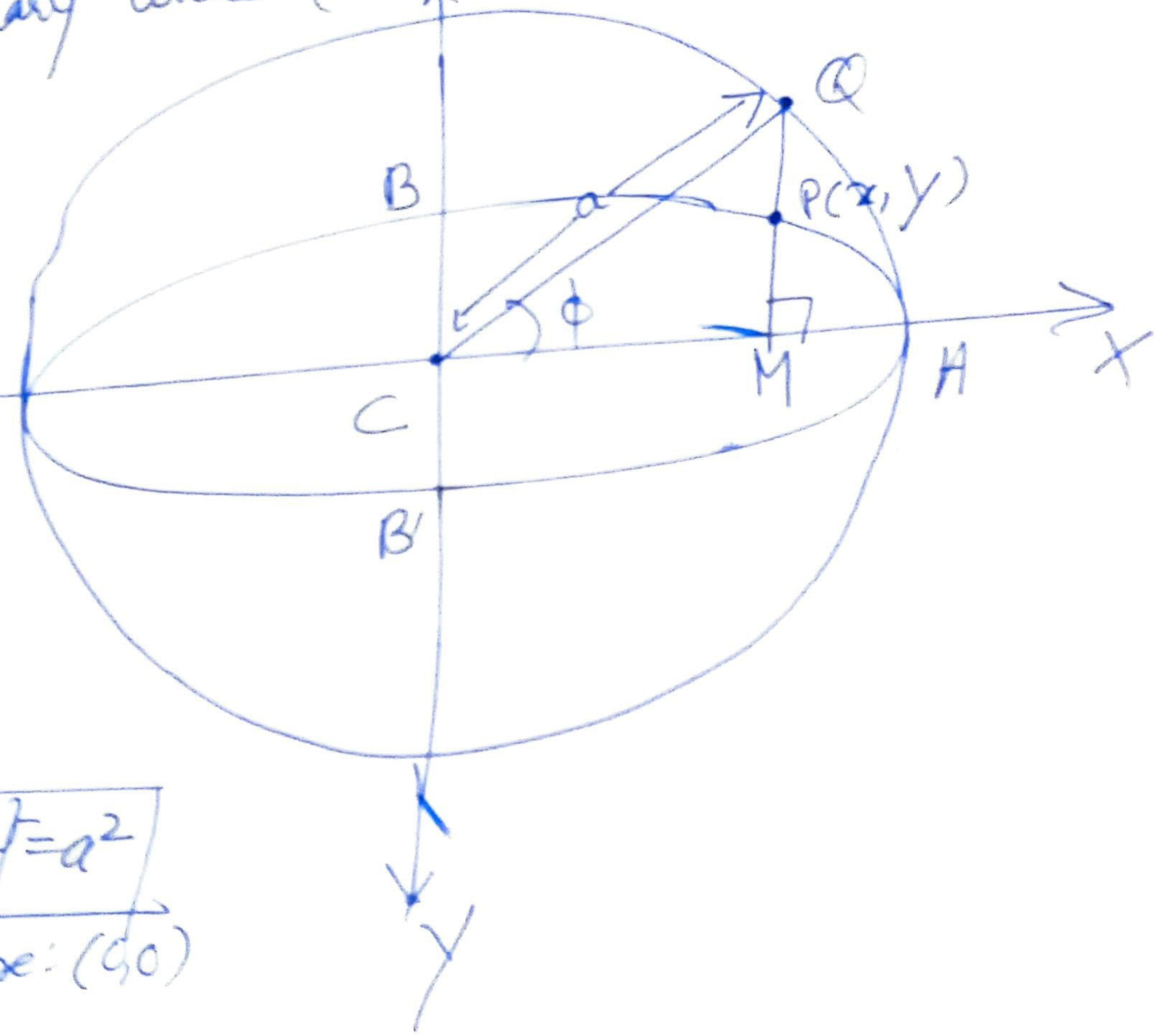
OR

2nd Form $$\boxed{\frac{a^2x}{x_1} - \frac{b^2y}{y_1} = a^2 - b^2}$$

Eqⁿ of Normal in Slope Form -

$$y = mx - \frac{m(a^2-b^2)}{\sqrt{a^2+b^2m^2}}$$

Auxiliary Circle - The circle with Major Axis AA' of the ellipse as diameter is called Auxiliary circle (सहायक वृत्त).



eqⁿ is

$$\boxed{x^2+y^2=a^2}$$

Centre: (0,0)

Radius = a

Eccentric Angle of a Point (बिंदु का उत्केंद्री कोण)

Let $P(x, y)$: any pt. on Ellipse

Drop perpendicular PM on Major Axis. Extend MP s.t. it will cut the Auxiliary Circle in Q. Join CQ.

Angle $\angle XCQ$: Eccentric angle of pt. P denoted by ϕ.

$$\cos\phi = \frac{x}{a} \Rightarrow x = a\cos\phi \quad \text{--- (1)}$$

Put in eqn of Ellipse

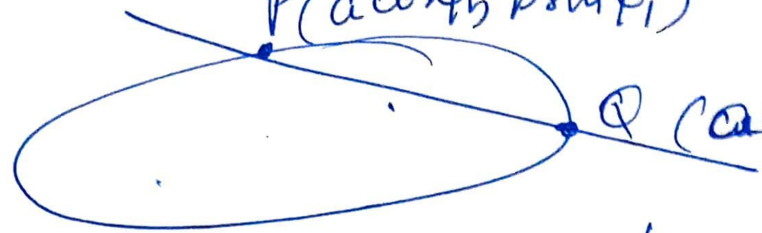
$$\frac{a^2\cos^2\phi}{a^2} + \frac{y^2}{b^2} = 1$$

$$\therefore y^2 = b^2(1-\cos^2\phi)$$

$$= b^2\sin^2\phi$$

$$\therefore y = b\sin\phi \quad \text{--- (2)}$$

$\therefore$ Parametric Coordinates (प्राचलिक निर्देशांक) of P: ~~(a cos~~ $\boxed{(a\cos\phi,\ b\sin\phi)}$



Parametric Coordinates

PQ: Chord (जीवा) of Ellipse

$$y - b\sin\phi_1 = \frac{b\sin\phi_2 - b\sin\phi_1}{a\cos\phi_2 - a\cos\phi_1}(x - a\cos\phi_1)$$

eqn:
$$\boxed{\frac{x}{a}\cos\left(\frac{\phi_1+\phi_2}{2}\right) + \left(\frac{y}{b}\right)\sin\left(\frac{\phi_1+\phi_2}{2}\right) = \cos\left(\frac{\phi_1-\phi_2}{2}\right)}$$

Eqn of Tangent in Parametric Form -

when $\phi_2 \rightarrow \phi_1$ then PQ touches
tends to

Ellipse & becomes Tangent

$$\boxed{\left(\frac{x}{a}\right)\cos\phi_1 + \left(\frac{y}{b}\right)\sin\phi_1 = 1}$$

Particular Case: when $b = a$ then

on circle $x^2+y^2=a^2$ at Pt. $(a\cos\phi, b\sin\phi)$

Tangent is $\boxed{x\cos\phi + y\sin\phi = a}$

Eqn of Normal in Parametric Form -

Normal is

$$\frac{x-x_1}{x_1/a^2} = \frac{y-y_1}{y_1/b^2}$$

At $(a\cos\phi, b\sin\phi)$

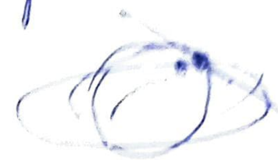
$$\frac{x-a\cos\phi}{a\cos\phi/a^2} = \frac{y-b\sin\phi}{b\sin\phi/b^2}$$

$$ax\sec\phi - a^2 = by\,\mathrm{cosec}\,\phi - b^2$$

$$\boxed{ax\sec\phi - by\,\mathrm{cosec}\,\phi = a^2-b^2}$$

उदाहरण .11.6 Ellipse(e)

Ex 4 A circle of radius r is concentric (~~समान~~ common centre) with the ellipse $x^2/a^2 + y^2/b^2 = 1$. Prove that the common tangent is inclined to the major axis at an angle $\tan^{-1}\left[\sqrt{\frac{r^2-b^2}{a^2-r^2}}\right]$ where $a > r > b$ are the semi axes of the ellipse. Find ~~its length also~~



Sol Ellipse: $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ — (1)

Concentric circle $x^2+y^2=r^2$ — (2)

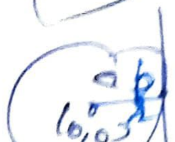
Tangent (स्पर्श रेखा) of ellipse in slope form (ढाल रूप)

(slope form is taken ∵ of presence of $m = \tan\theta$ ($\tan^{-1}$))

$$y = mx + \sqrt{a^2m^2+b^2}$$

or $mx - y + \sqrt{a^2m^2+b^2} = 0$ — (3)

Since line (3) touches circle (2)



∴ The length of perpendicular (लम्ब) from centre of circle (0,0) = radius of circle (property of tangent)

$\therefore$ Length of perp $= r = \dfrac{m \cdot 0 - 0 + \sqrt{a^2m^2+b^2}}{\sqrt{m^2+1}}$

$r\sqrt{m^2+1} = \sqrt{a^2m^2+b^2}$

Square both sides

$r^2(m^2+1) = a^2m^2+b^2$ or $r^2m^2 + r^2 = a^2m^2 + b^2$

or $m^2(a^2 - r^2) = r^2 - b^2$

$m^2 = \dfrac{r^2 - b^2}{a^2 - r^2} \Rightarrow m = \sqrt{\dfrac{r^2-b^2}{a^2-r^2}}$

Slope of (CT) $= m = \tan\phi = \sqrt{\dfrac{r^2-b^2}{a^2-r^2}}$

$\therefore \phi = \tan^{-1}\left[\sqrt{\dfrac{r^2-b^2}{a^2-r^2}}\right]$

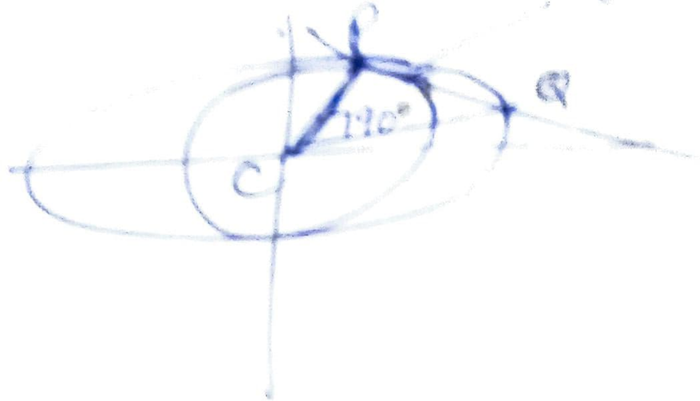
where ϕ is angle between common tangent & major axis.

EX3 If the portion of the line $x\cos\alpha + y\sin\alpha = p$ intercepted ... subtend a right angle (90°) at centre of ellipse prove that the line touches the circle of radius $ab/\sqrt{a^2+b^2}$ concentric with the ellipse.

Sol.

PQ touches ellipse
circle

(9)



ellipse $x^2/a^2 + y^2/b^2 = 1$ —— (1)

line $x\cos\alpha + y\sin\alpha = p$ —— (2)

To get joint eqn (संयुक्त समीकरण) of CP & CQ & make them homogeneous (समांगी) power 2

homogeneous => all variables have same powers

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \left(\frac{x\cos\alpha + y\sin\alpha}{p}\right)^2 \quad = 1$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{x^2\cos^2\alpha + 2xy\cos\alpha\sin\alpha + y^2\sin^2\alpha}{p^2}$$

$$\cancel{\frac{p^2}{a^2}x^2 + \frac{p^2y^2}{b^2} - x^2\cos^2\alpha - y^2\sin^2\alpha - 2xy\cos\alpha\sin\alpha = 0}$$

or $$\frac{x^2}{a^2} - \frac{x^2\cos^2\alpha}{p^2} - \frac{2xy\cos\alpha\sin\alpha}{p^2} + \frac{y^2}{b^2} - \frac{y^2\sin^2\alpha}{p^2} = 0$$

$$x^2\left[\frac{1}{a^2} - \frac{\cos^2\alpha}{p^2}\right] - 2\frac{\sin\alpha\cos\alpha}{p^2}xy + y^2\left[\frac{1}{b^2} - \frac{\sin^2\alpha}{p^2}\right] = 0 \quad \text{—— (3)}$$

In (3), CQ' will be at 90° to CQ if coefficient of x^2 + coeff of $y^2 = 0$

([illegible])

or $\left(\frac{1}{a^2} - \frac{\cos^2\alpha}{p^2}\right) + \left(\frac{1}{b^2} - \frac{\sin^2\alpha}{b^2}\right) = 0$

or $\frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{p^2}(\cos^2\alpha + \sin^2\alpha)$

or $\frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{p^2}$ ——— (4)

Eqn of concentric circle with centre (0,0) & radius $\frac{ab}{\sqrt{a^2+b^2}}$ (given in question) is

$x^2+y^2 = \frac{a^2b^2}{a^2+b^2}$ ——— (5)

if line (2) touches circle (5) then length of perp from centre (0,0) = radius of circle

$\frac{0.\cos\alpha + 0.\sin\alpha - p}{\sqrt{\cos^2\alpha+\sin^2\alpha}} = \sqrt{\frac{a^2b^2}{a^2+b^2}}$

Squaring both sides

Or $\frac{p^2}{1} = \frac{a^2b^2}{a^2+b^2}$ ~~divide by a^2b^2~~

or $\frac{1}{p^2} = \frac{a^2+b^2}{a^2b^2} = \frac{1}{b^2} + \frac{1}{a^2} = (4)$

([illegible]) Hence Proved.