Director Circle - The locus of the pt. of intersection of 2 perpendicular tangents of an ellipse is called its Director Circle.

$$\boxed{x^2+y^2=a^2+b^2}$$

Let (x_1, y_1) : pt. of intersection of 2 perpendicular tangents,

$\because$ joint eq^n of Tangents $\qquad$ $(SS_1 = T^2)$ eq of pair of Tangents

$$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)\left(\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1\right)=\left(\frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1\right)^2$$

If these 2 tangents are at $90°$, then

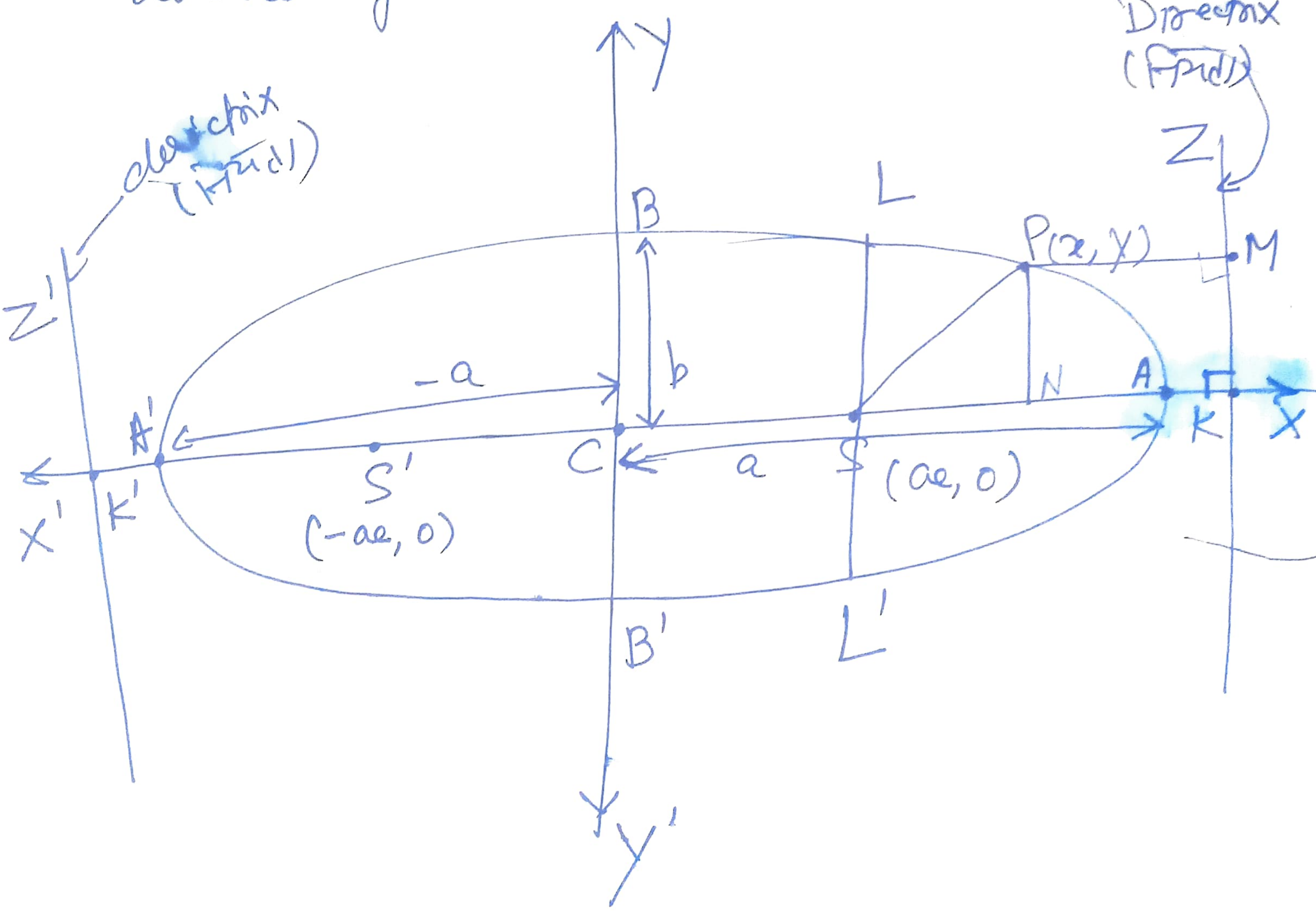
Coefficient of x^2 + Coefficient of $y^2 = 0$

$$\therefore \left[\frac{1}{a^2}\left(\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1\right)-\frac{x_1^2}{a^4}\right]+\left[\frac{1}{b^2}\left(\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1\right)-\frac{y_1^2}{b^4}\right]=0$$

Or $\dfrac{y_1^2}{a^2b^2}-\dfrac{1}{a^2}+\dfrac{x_1^2}{a^2b^2}-\dfrac{1}{b^2}=0$ Or $x_1^2+y_1^2=a^2+b^2$

$\therefore$ Locus of (x_1, y_1) : $\boxed{x^2+y^2=a^2+b^2}$ — Director Circle

Ellipse (दीर्घवृत्त) – It is the locus (बिंदुपथ) of a point which moves such that (s.t.) its distance from a fixed point (स्थिर बिंदु) called <u>focus</u> (नाभि) to its distance from a <u>fixed straight line</u> called <u>directrix (नियता)</u> bears a constant ratio called <u>eccentricity (उत्केन्द्रता)</u> which is always less than 1. The eccentricity is denoted by e.



Focal Property of Ellipse —

Ellipse: $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

Focii: $(ae,0)$, $(-ae,0)$, $CS'=CS=ae$
$S \,\&\, S'$

and $CK'=CK=\frac{a}{e}$

$SP=e\cdot PM = e\cdot NK = e(CK-CN)$

$= e\left(\frac{a}{e}-x\right)=a-ex$

$S'P = e\cdot PM' = e\cdot(NC+CK')$

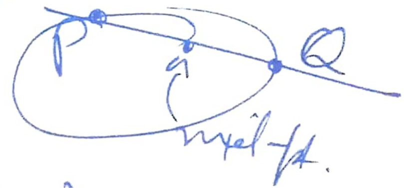
$= e\left(x+\frac{a}{e}\right) = a+ex$

$\therefore SP+S'P = 2a$ (constant) $=$ Major Axis

2nd Defn of Ellipse — An ellipse is the locus of a pt. which moves s.t. the sum of its distances from 2 fixed pts. (focii) is constant.

Eq^n of Chord of Ellipse whose mid-pt. is given—

दीर्घवृत्त की जीवा का समीकरण जिसका मध्यबिंदु दिया है—



Let P be (x_2, y_2) & Q is (x_3, y_3)

mid-pt. of chord is (x_1, y_1) where

$$x_1 = \frac{x_2 + x_3}{2} \quad \text{—(1)} \qquad y_1 = \frac{y_2 + y_3}{2} \quad \text{—(2)}$$

Since P & Q lie on the Ellipse $\therefore \frac{x_2^2}{a^2} + \frac{y_2^2}{b^2} = 1$ —(3)

$$\& \ \frac{x_3^2}{a^2} + \frac{y_3^2}{b^2} = 1 \quad \text{—(4)}$$

(4) − (3) ⇒

$$\frac{1}{a^2}(x_3^2 - x_2^2) + \frac{1}{b^2}(y_3^2 - y_2^2) = 0$$

or $b^2(x_3^2 - x_2^2) + a^2(y_3^2 - y_2^2) = 0$

or $a^2(y_3 - y_2)(y_3 + y_2) + b^2(x_3 - x_2)(x_3 + x_2) = 0$

or $a^2(y_3 - y_2)(2y_1) + b^2(x_3 - x_2)(2x_1) = 0$

(from (1) & (2))

Or $\dfrac{y_3-y_2}{x_3-x_2} = -\dfrac{b^2x_1}{a^2y_1}$ —(5)

Gradient (Slope) of PQ $= \dfrac{y_3-y_2}{x_3-x_2} = \dfrac{-b^2x_1}{a^2y_1}$

PQ passes thru (x_1, y_1)

∴ its eq^n: $y-y_1 = \dfrac{-b^2x_1}{a^2y_1}(x-x_1)$

$a^2yy_1 - a^2y_1^2 = -b^2xx_1 + b^2x_1^2$

$xx_1b^2 + yy_1a^2 = a^2y_1^2 + b^2x_1^2$

Divide by $a^2b^2 \Rightarrow$

$\dfrac{xx_1}{a^2} + \dfrac{yy_1}{b^2} = \dfrac{x_1^2}{a^2} + \dfrac{y_1^2}{b^2}$

or $\dfrac{xx_1}{a^2} + \dfrac{yy_1}{b^2} - 1 = \dfrac{x_1^2}{a^2} + \dfrac{y_1^2}{b^2} - 1$

or $\boxed{T = S_1}$

eq^n of Tangent

E11 (c)

Diameter (व्यास) – Locus of mid-pts of a system of Parallel chords of an ellipse is diameter of Ellipse.

Let Parallel chords be: $y = mx = mx + c$

Let $y = mx + c$ is eqn of chord out of these 2 parallel chords.

Put in Ellipse ⇒

$$\frac{x^2}{a^2} + \frac{(mx+c)^2}{b^2} = 1$$

$$b^2x^2 + a^2(m^2x^2 + 2mcx + c^2) = a^2b^2$$

Or $(a^2m^2 + b^2)x^2 + 2a^2mcx + (c^2 - b^2)a^2 = 0$

Quadratic in x ($Ax^2 + Bx + C = 0$, Sum of Roots $= -B/A$)

Sum of Roots

$$x_1 + x_2 = \frac{-2a^2mc}{a^2m^2 + b^2}$$

Let (h, k) be the mid-pt. of PQ ⇒

$$h = \frac{x_1 + x_2}{2} = \frac{-a^2mc}{a^2m^2 + b^2} \quad \text{—(1)}$$

Or $$c = \frac{-h(a^2m^2 + b^2)}{a^2m}$$

But (h, k) lies on PQ

$$\therefore k = mh + c$$

$$c = k - mh$$

Put value of c in (1) $\Rightarrow$

$$h = \frac{-a^2 m (k - mh)}{a^2 m^2 + b^2}$$

$$\cancel{a^2 m^2 h} + b^2 h = -a^2 mk + \cancel{a^2 m^2 h}$$

$$b^2 h = -a^2 mk \quad \text{or} \quad k = -\frac{b^2}{a^2 m} h$$

$\therefore$ Locus of mid pt. of PQ is

$$\boxed{y = -\frac{b^2}{a^2 m} x}$$ — eqn of Diameter

(संयुग्मी व्यास)

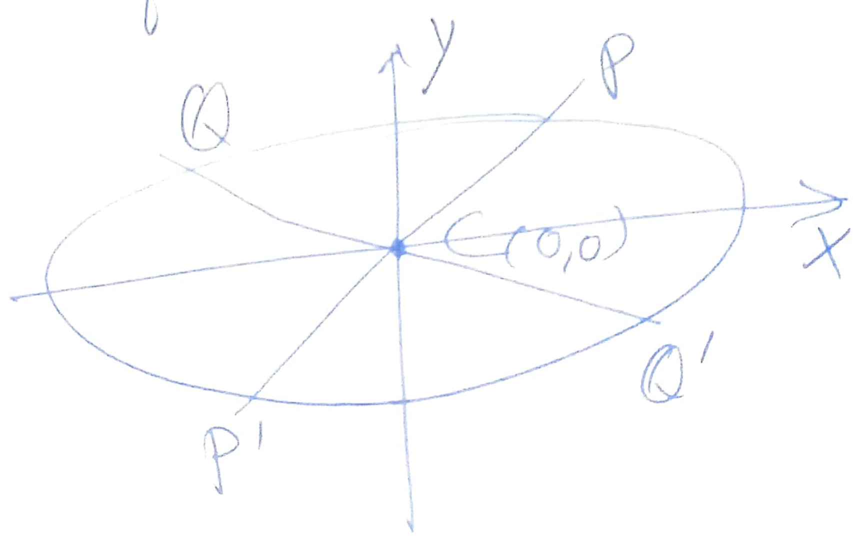
<u>Conjugate Diameters</u> – 2 diameters are conjugate to each other when each bisects the chord parallel to other.

The Condition for the lines $y = mx$ & $y = m'x$ to be conjugate diameters wrt ellipse is : $mm' = -\frac{b^2}{a^2}$ — (1)

Note: If $y = mx$ is a diameter of Ellipse then its conjugate diameter is $y = -\frac{b^2}{a^2 m} x$.

Properties of Conjugate Diameters –

(1) The eccentric angles of extremities (ecc. α or end points) of 2 conjugate semi-diameters of an ellipse differ by 90°



CP & CQ are conjugate semidiameters

Let $P(a\cos\theta, b\sin\theta)$,

$Q(a\cos\phi, b\sin\phi)$

Gradient of $CP = \dfrac{b\sin\phi - 0}{a\cos\phi - 0} = m_1$

Gradient of $CQ = \dfrac{b\sin\phi - 0}{a\cos\phi - 0} = m_2$

CP & CQ are conjugate semidiameters

$\Rightarrow m_1 m_2 = -\dfrac{b^2}{a^2}$ [from the condition]

$$\frac{b^2\sin\theta\sin\phi}{a^2\cos\theta\cos\phi} = -\frac{b^2}{a^2}$$

or $\dfrac{\sin\theta\sin\phi}{\cos\theta\cos\phi} = -1$

or $\cos\theta\cos\phi + \sin\theta\sin\phi = 0$

or $\cos(\theta - \phi) = \cos\left(\dfrac{\pi}{2}\right)$

$\theta - \phi = \dfrac{\pi}{2}$ HP.

$\therefore$ If P & Q are extremities of conjugate semidiameters & if P is $(a\cos\theta, b\sin\theta)$ then $\boxed{\theta = \frac{\pi}{2} + \phi}$

then Q is $(-a\sin\theta, b\cos\theta)$

[sin এ negative হয় যাবে] in the Q, বাম quadrant এ

য়।

2.) Sum of squares of conjugate semi-diameters of Ellipse is constant & equal to sum of squares of semi axes of Ellipse, $CP^2 + CQ^2 = a^2 + b^2$

Sum of squares of conjugate semi-diameters

$= CP^2 + CQ^2 = \left[(a\cos\theta - 0)^2 + (b\sin\theta - 0)^2\right]$

$+ \left[(-a\sin\theta - 0)^2 + (b\cos\theta - 0)^2\right]$

$= a^2\cos^2\theta + b^2\sin^2\theta + a^2\sin^2\theta + b^2\cos^2\theta = a^2 + b^2$

HP.

(3.) Tangents drawn at ends (extra) of any diameter of Ellipse is parallel to conjugate diameter.

Let PCP' & QCQ' : Conjugate diameters

eqn of Tangent at P of PCP' is: $\frac{x \cancel{a}\cos\theta}{a^{\cancel{2}}} + \frac{y \cancel{b}\sin\theta}{b^{\cancel{2}}} = 1$

$\frac{y\sin\theta}{b} = \frac{-x\cos\theta}{a} + 1$

or $y = -\frac{b}{a}\frac{\cos\theta}{\sin\theta}x + \frac{b}{\sin\theta}$

Slope $= \frac{-b}{a}\frac{\cos\theta}{\sin\theta} = \frac{-b}{a}\cot\theta$

$(\because y = mx + c)$

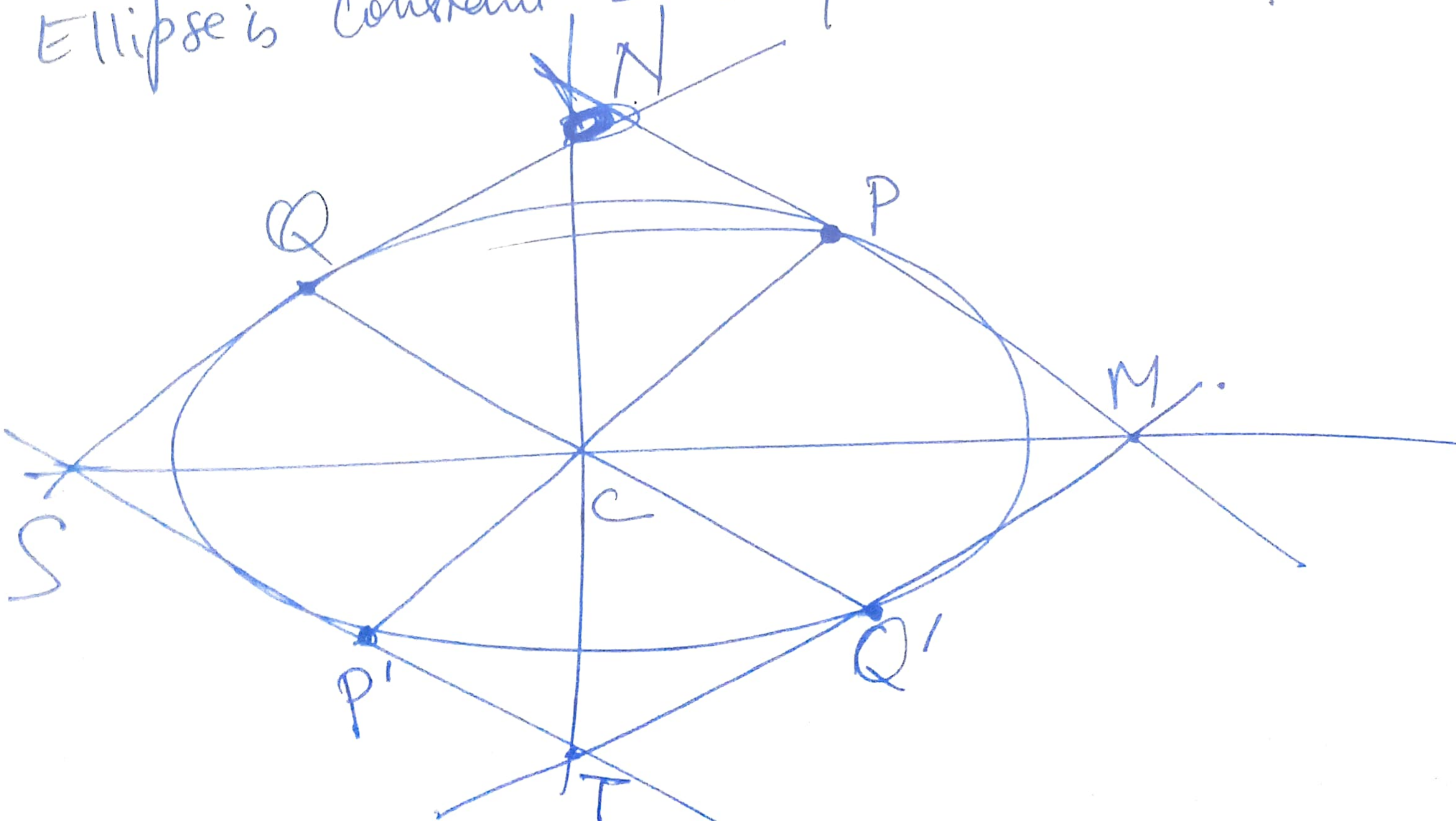
Gradient of $QCQ' = \frac{b\cos\theta - 0}{-a\sin\theta - 0} = -\frac{b}{a}\cot\theta$

$Q(-a\sin\theta, b\cos\theta)$

$C : (0, 0)$

Both slopes are same i.e., Tangent at P is parallel to QCQ' which is conjugate diameter of PCP'. Similarly Tangent at Q is parallel to RP'

(4) Area of Parallelogram formed by Tangent at ends of conjugate diameter of an Ellipse is constant & is equal to Product of Axes



E11 (i)

Area of MNST = 4 × Area NCPQ

where PCP' & QCQ' : conjugate diameter

P(a cos θ, b sin θ), Q(−a sin θ, b cos θ)

∴ Area of MNST = 8× Area (ΔPNC)

$$= 8\times\frac{1}{2}\begin{vmatrix} a\cos\theta & b\sin\theta & 1 \\ -a\sin\theta & b\cos\theta & 1 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= 4\left(ab\cos^2\theta + ab\sin^2\theta\right)$$

$$= 4ab$$

$$= 2a \times 2b$$

= Product of major Minor axes.

2a 2b.

length, width/breadth

In 2-D geometry Curves are - Circle, Ellipse, Parabola, Hyperbola etc.

3-D : Depth, breadth, height

~~Conic Section or Conic (शंकु-परिच्छेद या शांकव)~~

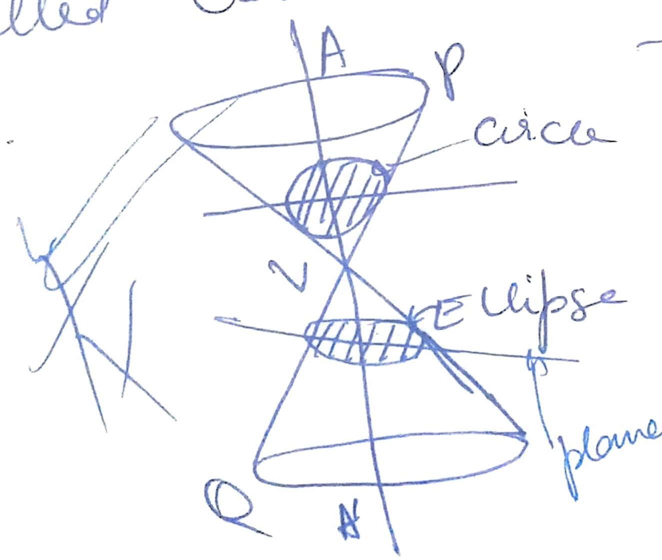
parallelogram

Above mentioned all curves can be represented by a general eqn of 2nd degree (व्यापक द्विघात समीकरण):

$$ax^2+2hxy+by^2+2gx+2fy+c=0=f(x,y)$$

Conic Section or Conic (शंकु-परिच्छेद या शांकव)

When a circular Cone (वृत्तीय शंकु) is cut by a plane (समतल) in various ways then the sections (परिच्छेद) obtained are called Conic Sections (शंकु-परिच्छेद).



Defn : A Conic section or Conic is the locus of a pt. P which moves so that its distance from a fixed pt. S (focus-नाभि) is in a constt ratio to its perpendicular (90°) distance from a fixed straight line (directrix-नियता), all being in same plane.

Gorakhpur P38

Ex. 13 Trace the foll. curve

$$x^2-3xy+y^2+10x-10y+21=0$$

Sol.

$$ax^2+2hxy+by^2+2gx+2fy+c=0$$

$$a=1,\ h=-\tfrac{3}{2},\ b=1,\ g=5,\ f=-5,\ c=21$$

$$\Delta = abc+2fgh-af^2-bg^2-ch^2$$

$$=21+\cancel{2}(-5)(5)\left(-\tfrac{3}{\cancel{2}}\right)-1\cdot(-5)^2-1(5)^2-(21)\left(-\tfrac{3}{2}\right)^2$$

$$=21+75-25-25-\frac{189}{4}=-5/4\neq 0$$

$$h^2-ab=\left(\tfrac{9}{4}\right)-1=5/4>0$$

$\therefore$ Conic is Hyperbola

$$\phi(x,y)=x^2-3xy+y^2+10x-10y+21=0$$

To find Centre

$$\frac{\partial\phi}{\partial x}=0 \Rightarrow 2x-3y+10=0 \quad \text{—— (2)}$$

$$\frac{\partial\phi}{\partial y}=0 \Rightarrow \cancel{2y-3x+10=0}\ \ 3x-2y+10=0 \quad \text{—— (3)}$$

$(2)\times 3-(3)\times 2 \Rightarrow$

$$6x-9y+30=0$$
$$\underline{6x-4y+20=0}$$
$$-5y=-10,\quad y=2$$

$$\cancel{6x-9y+30=0}$$
$$\underline{-6x+6y+20=0}$$
$$-5y=-10$$

$\therefore 2x-3\times2+10=0$

$2x=-2$ $\therefore$ Centre $(-2,2)$

Centre: $(-2, 2)$

Eqn of Conic on taking centre as origin

$$ax^2 + 2hxy + by^2 + \frac{\Delta}{ab-h^2} = 0$$

$$x^2 - 3xy + y^2 + \frac{(-5/4)}{(-5/4)} = 0$$

$$x^2 - 3xy + y^2 = -1$$

In standard Form, $-x^2 + 3xy - y^2 = 1$

$$A = -1,\quad H = \frac{3}{2},\quad B = -1$$

Lengths of semi-axes will be roots of the foll. eqn:

$$(AB - H^2)r^4 - (A+B)r^2 + 1 = 0$$

$$\left[1 - \frac{9}{4}\right] r^4 - [-1-1] r^2 + 1 = 0$$

$$\left(\frac{4-9}{4}\right) \left[\frac{-6-9}{4}\right] r^4 + 2r^2 + 1 = 0 \quad \text{Or} \quad -\frac{5}{4} r^4 + 2r^2 + 1 = 0$$

$$\text{Or} \quad -5r^4 + 8r^2 + 4 = 0 \quad \text{Or} \quad 5r^4 - 8r^2 - 4 = 0$$

$$\left(\frac{4-9}{4}\right)x^4+2x^2+1=0$$

$$-5x^4+8x^2+4=0$$

$$5x^4-8x^2-4=0$$

$5x^4-10x^2+2x^2-4=0$ or $\cancel{5x^2(x^2-2)}\ (x^2-2)=0$

$\cancel{(x^2-2)(5x^2}$ $\quad 5x^2(x^2-2)+2(x^2-2)=0$

$$(x^2-2)(5x^2+2)=0$$

$$x_1^2=2,\quad x_2^2=-\frac{2}{5}$$

Eqn of axes

$\left(A-\frac{1}{x_1^2}\right)x+Hy=0$ or $\left(-1-\frac{1}{2}\right)x+\frac{3}{2}y=0$

$-\frac{3}{2}x+\frac{3}{2}y=0$ or $-x+y=0$ or $x-y=0$

(+ve Root $r_1 \Rightarrow x-y=0$ is Transverse Axis)

$\left(A-\frac{1}{x_2^2}\right)x+Hy=0$ or $\left(-1+\frac{2}{5}\right)x+\frac{3}{2}y=0$

or $x+y=0$

(−ve root $r_2 \Rightarrow x+y=0$ is Conjugate Axis) मूल अक्षों को अक्ष मानकर

Axis on ~~taking~~

Eqn of Transverse Axis ~~w.r.t. the Coordinate axes~~ $\Rightarrow$

(~~Conf. is focus~~) as the

$x-(-2)-(y-2)=0$

Or $(x+2)-(y-2)=0$ or $x-y+4=0$ Axis.

Conjugate Axis does not cut the Y-Axis in Real points.

To Hyp (4)

Tracing

1. 2 perpendicular lines XOX' & YOY'

2. $C(-2, 2)$ - Centre

3. $CA = A'C = \sqrt{2}$

4. Perpendicular : from A to ACA'.

5. On this, $D'A = AD = \sqrt{2/5}$

put $y=0$ put $x=0$

6. points $(-3,0), (-7,0), (0,3), (0,7)$ are points of intersection of Hyp., & the axes

Pl. see diagram in Gorakhpur's book P39