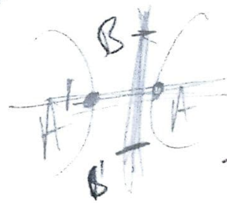
Hyperbola

BB' : conjugate Axis



AA' : Transverse Axis

Eq^n of pair of Tangents

$$SS_1 = T^2$$

$$\left(\frac{x^2}{a^2} - \frac{y^2}{b^2} - 1\right)\left(\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1\right) = \left(\frac{xx_1}{a^2} + \frac{yy_1}{b^2} - 1\right)^2$$

Director Circle

$$x^2 + y^2 = a^2 - b^2$$

Eq^n of Chord of Contact of Tangent

from (x_1, y_1) on $x^2/a^2 - y^2/b^2 = 1$

$$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$$

Eq^n of Polar — Pole (x_1, y_1)

$$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$$

Eq^n of chord whose middle pt. is given

mid pt (x_1, y_1) , $T = S_1$

$$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2}$$

Diameter: locus of mid pts of chords which are parallel to $y = mx$ Hyp(f)

eqn of diameter: $y = \frac{b^2}{a^2 m} x$

Conjugate diameter -

$y = m_1 x$ & $y = m_2 x$ are conjugate diameters to $x^2/a^2 - y^2/b^2 = 1$ if $m_1 m_2 = b^2/a^2$

Auxiliary Circle -

$x^2 + y^2 = a^2$

This circle is drawn with Transverse Ax. (अनुप्रस्थ अक्ष) as diameter.

U.b

EX. 3 Prove that the locus of poles of chords of the hyperbola $x^2/a^2 - y^2/b^2 = 1$ which subtend a right angle at the centre is

$$\frac{x^2}{a^4} + \frac{y^2}{b^4} = \frac{1}{a^2} - \frac{1}{b^2}.$$

Sol. Let Pole be (x_1, y_1) of ~~other~~ chord of hyperbola $x^2/a^2 - y^2/b^2 = 1$ then Polar (ie, chord) is

$$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1 \quad -(1)$$

Joint eq^n of lines joining hyperbola & chord (1) making homogeneous

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = \left(\frac{xx_1}{a^2} - \frac{yy_1}{b^2}\right)^2$$

Or $\left(\frac{xx_1}{a^2} - \frac{yy_1}{b^2}\right)^2 = \frac{x^2}{a^2} - \frac{y^2}{b^2}$

Hyp (h)

or $\frac{x^2 x_1^2}{a^4} + \frac{y^2 y_1^2}{b^4} - \frac{2xy x_1 y_1}{a^2 b^2} = \frac{x^2}{a^2} - \frac{y^2}{b^2}$

or $\left(\frac{x_1^2}{a^4} - \frac{1}{a^2}\right)x^2 + \left(\frac{y_1^2}{b^4} + \frac{1}{b^2}\right)y^2 - \frac{2x_1 y_1 xy}{a^2 b^2} = 0$ —(2)

The lines in (2) are perpendicular

$\therefore$ Coeff. of x^2 + coefficient of $y^2 = 0$
(গুণাংক)

$$\left(\frac{x_1^2}{a^4} - \frac{1}{a^2}\right) + \left(\frac{y_1^2}{b^4} + \frac{1}{b^2}\right) = 0$$

$$\frac{x_1^2}{a^4} + \frac{y_1^2}{b^4} = \frac{1}{a^2} - \frac{1}{b^2}$$

$\therefore$ locus of (x_1, y_1):

$$\frac{x^2}{a^4} + \frac{y^2}{b^4} = \frac{1}{a^2} - \frac{1}{b^2}$$

1+0

II C P87, [illegible] Hyp 4

2. Prove that Polar of any pt. on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ wrt (with respect to) Hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ touches the ellipse.

(pt. upon ellipse)
pt. is pole

Sol. Let (α,β) be pt. on ellipse then

$$\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}=1 \quad -(1)$$

Polar of (α,β) wrt Hyper: $\frac{\alpha x}{a^2}-\frac{\beta y}{b^2}=1$

or $\frac{y\beta}{b^2}=\frac{\alpha x}{a^2}-1 \quad \therefore y=\frac{b^2}{\beta}\left(\frac{\alpha x}{a^2}-1\right)$

$(y=mx+c)$

or $y=\left(\frac{b^2\alpha}{\beta a^2}\right)x-\frac{b^2}{\beta}$

Here $m=\frac{b^2\alpha}{a^2\beta}$, $\quad c=-\frac{b^2}{\beta}$

We know $y=mx+c$ touches ellipse

if $c^2=a^2m^2+b^2$

$\left(\frac{b^2}{\beta}\right)^2=a^2\left(\frac{b^2\alpha}{a^2\beta}\right)^2+b^2$ or $\frac{b^4}{\beta^2}=\frac{a^2b^4\alpha^2}{a^4\beta^2}+b^2$

divide by b^4

or $\frac{1}{\beta^2}=\frac{\alpha^2}{a^2\beta^2}+\frac{1}{b^2}$, Multiply by β^2

$\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}=1$ which is (1)

Ex Find eqn of Tangent of Hyperbola $x^2-y^2=\left(\frac{1}{2}\right)^2$ which is parallel to the line $4y=5x+7$.

Sol. Line parallel to line $4y=5x+7$ is

$4y=5x+k$ —— (1) k: constant to be determined

If (1) touches Hyperbola then.

$$x^2-\left(\frac{5x+k}{4}\right)^2=\frac{1}{4}$$

or $16x^2-(25x^2+10kx+k^2)=4$

or $9x^2+10kx+(k^2+4)=0$ —— (2)

For Tangent, roots of (2) should be equal

or Discriminant of this quadratic eqn $=0$

$100k^2=4\times9\times(k^2+4)$ ($Ax^2+Bx+C=0$, $D=B^2-4AC$)

$\therefore$ eqn is $100k^2=36k^2+144$

or $k^2=9/4=3/2$

$\therefore$ Tangent eqn: $4y=5x+3/2$ Ans.

eq^n of tangent & Normal at $r.H.P$ (a) (x_1, y_1) to the rectangular hyperbola $xy = c^2$ is

Tangent: $\frac{x}{x_1} + \frac{y}{y_1} = 2$

Normal: $xx_1 - yy_1 = x_1^2 - y_1^2$

Polar: $xy_1 + yx_1 = 2c^2$

Parametric Coords -

$x = ct$ & $y = c/t$ for every value of t satisfies $xy = c^2$ $\therefore$ Parametric Coords of hyperbola $xy = c^2$ are $(ct, c/t)$

Tangent: $x/t + yt = 2c$

Normal: $xt^3 - yt - ct^4 + c = 0$

Hyp i

EX. 7 (P77): If $x=2y$ is one of the diameters of the hyperbola $16x^2-9y^2=144$, find eqn of its conjugate diameter.

$\frac{16x^2}{144}-\frac{9y^2}{144}=1$ or $\frac{x^2}{9}-\frac{y^2}{16}=1$

Sol. $a^2=9,\ b^2=16$

One diameter is $x=2y$ or $y=\frac{1}{2}x$

$m_1 = 1/2$ (from $2y=x$, $y=\frac{x}{2}$)

(प्रवणता)

Conjugate diameter condition

$m_1 m_2 = b^2/a^2$

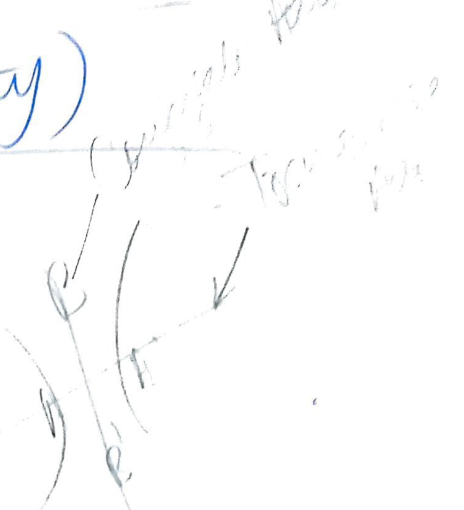
$\therefore m_2 = b^2/a^2 \times \frac{1}{m_1} \Rightarrow M_2 = \frac{16}{9}\times 2x$

or $y=(32/9)x$ Ans

Hyp (j)

Asymptote: (अनंतस्पर्शी) A straight line at a finite distance from the origin to which the tangent to a curve tends as the pt. of contact tends to infinity is called an asymptote of the curve.

i.e., (Tangent at infinity)



Asymptotes are:

$$\frac{x}{a}-\frac{y}{b}=0 \quad \& \quad \frac{x}{a}+\frac{y}{b}=0$$

joint eq^n of these:

$$\left(\frac{x}{a}-\frac{y}{b}\right)\left(\frac{x}{a}+\frac{y}{b}\right)=0 \Rightarrow \frac{x^2}{a^2}-\frac{y^2}{b^2}=0$$

1) There is difference of only one term in the joint eq^n of asymptote and eq^n of Hyperbola.

2) Asymptotes make the angle $\alpha = \tan^{-1} b/a$ with Transverse axis (अनंतस्पर्शी/अक्ष)

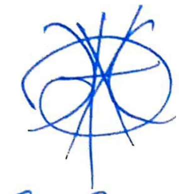
3) Angle between asymptotes:

$$= 2\tan^{-1} b/a \quad \text{or} \quad \tan^{-1}\frac{2ab}{a^2-b^2}$$

4) If $a=b$ i.e. Rectangular Hyperbola then asymptotes are $y=x$ & $y=-x$ i.e. at 90° to each other

P84 Hyp (0)

EX.1. Find the eqn of the hyperbola whose asymptotes are $3x-4y+7=0$ & $4x+3y+1=0$ & passes then the origin



Sol. There is difference of only one (3-121x 2111x1) between constant term the eqn of hyperbola $(x^2/a^2-y^2/b^2=1)$ & the joint eqn of asymptotes $\left(\frac{x^2}{a^2}-\frac{y^2}{b^2}=0\right)$

$\therefore$ let eqn of hyperbola be

$$(3x-4y+7)(4x+3y+1)+k=0 \quad -(1)$$

$\leftarrow$ constant

It passes then origin (मूल बिंदु) $(0,0)$

$$\therefore (0-0+7)(0+0+1)+k=0$$

$$\Rightarrow k=-7$$

$\therefore$ eqn of hyperbola is

$$(3x-4y+7)(4x+3y+1)-7=0$$

or $12x^2-7xy-12y^2+31x+17y=0$

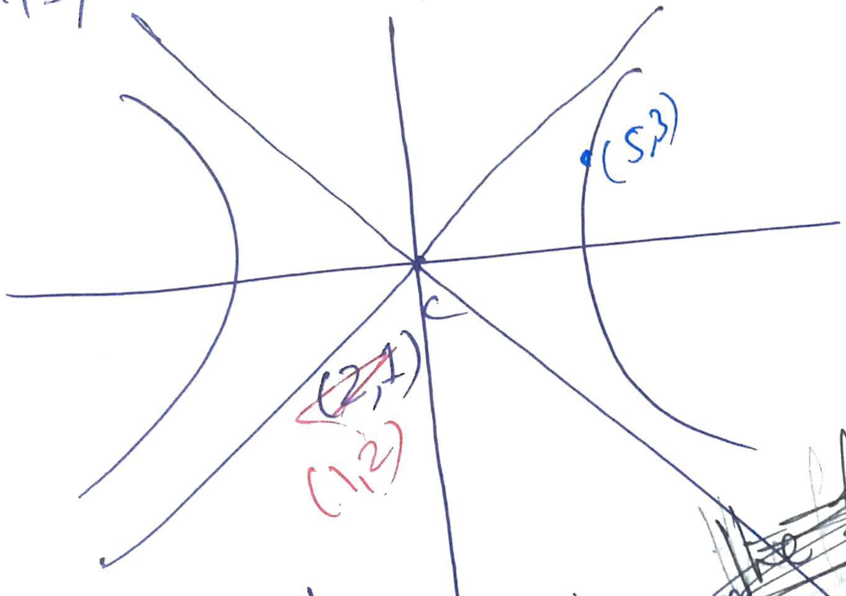
Ans.

Ex. Centre of Hyperbola is ~~(2,1)~~ (1,2). (Hyp 2)

Its asymptotes are parallel to $2x+3y=0$ & $3x+2y=0$. If the Hyperb passes thru $(5,3)$, show that its eqn is (Hyperbola's)

$(2x+3y-8)(3x+2y-7)=154$.

Sol.



Asymptotes are parallel to ~~the following given asymptotes~~ $2x+3y=0$ & $3x+2y=0$ —(1)

and are → $2x+3y+k_1=0$ & $3x+2y+k_2=0$ —(2)

(parallel lines have same slopes)

k_1, k_2: constants

Both pass thru ~~(2,1)~~ (1,2).

Put (1,2) in (1) & (2) ⇒

$2\times1+3\times2+k_1=0 \Rightarrow k_1=-8$

$3\times1+2\times2+k_2=0 \Rightarrow k_2=-7$

Hyp S

Asymptotes are

$(2x+3y-8)(3x+2y-7)=0$ ——— (3)

Hyperbola: $(2x+3y-8)(3x+2y-7)+k=0$ — (4)

($\because$ eqn of asymptotes & hyperbola differ only in one constant term)

(4) passes thru $(5,3)$

$\Rightarrow$

$[2\times5+3\times3-8]\,[3\times5+2\times3-7]+k=0$

$(10+9-8)(15+6-7)+k=0$

$(11)(14)+k=0 \quad \therefore k=-154$

$\therefore$ Put $k=-154$ in (4) we get the required eqn

$(2x+3y-8)(3x+2y-7)=154$ Ans.

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(Hyper)

4. Prove that product of perpendiculars from any point on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ to its asymptotes is

Constant & $=\frac{a^2b^2}{a^2+b^2}$.

Sol. Asymptotes (for standard Hyperbola) are

$\frac{x}{a}\pm\frac{y}{b}=0$ or $\frac{x}{a}+\frac{y}{b}=0$ & $\frac{x}{a}-\frac{y}{b}=0$

Let p_1 & p_2 be lengths of perpendiculars for a point (α,β) on the Hyperb. to its asymptotes : (i.e, perp on the asymptotes)

$$p_1\cdot p_2=\frac{\frac{\alpha\cdot 1}{a}+\frac{\beta\cdot 1}{b}}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}}}\cdot\frac{\frac{\alpha\cdot 1}{a}-\frac{\beta\cdot 1}{b}}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}}}=\frac{\frac{\alpha^2}{a^2}-\frac{\beta^2}{b^2}}{\frac{1}{a^2}+\frac{1}{b^2}}=\frac{1}{\frac{1}{a^2}+\frac{1}{b^2}}$$

($\because$ (α,β) lies on Hyperbola so satisfies its eqn) so $\frac{\alpha^2}{a^2}-\frac{\beta^2}{b^2}=1$)

$$=\frac{1}{\frac{a^2+b^2}{a^2b^2}}=\frac{a^2b^2}{a^2+b^2}$$ H.P.