Lattice - A lattice is a poset $(L, \leq)$ in which every subset $\{a, b\}$ of 2 elements of L has an LUB and a GLB. ie, $\forall a, b \in L$, $\sup\{a, b\}$ & $\inf\{a, b\}$ exist in L.

We denote $LUB\{a, b\}$ by $a \vee b$ & call it the join of a & b and denote $\inf\{a, b\}$ by $a \wedge b$ & call it the meet of a & b. Sometimes $a+b$ or $a \cup b$ and $a \cdot b$ & $a \cap b$ are also used for $LUB\{a, b\}$ & $GLB\{a, b\}$ respectively

ex. Let S be a non-empty set & $P(S)$: Power set. Then $(P(S), \subseteq)$ is a Lattice.

Sol. Let any 2 sets $A, B \in P(S)$

$A \vee B = LUB\{A, B\} = A \cup B$

$A \wedge B = GLB\{A, B\} = A \cap B$

Since both $A \cup B$ & $A \cap B \in P(S)$ $\therefore (P(S), \subseteq)$ is a Lattice.

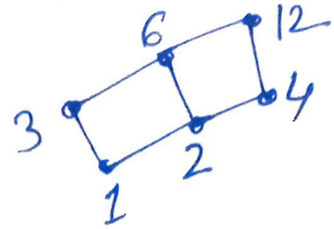
ex. Let L of all factors of 12 under divisibility forms a lattice

Sol. $L = \{1, 2, 3, 4, 6, 12\}$, let a, b be any 2 elts of L

$LUB\{a, b\} \Rightarrow$ the least positive integer c s.t. $a|c$ and $b|c \Rightarrow LUB\{a, b\} = LCM\{a, b\}$.

Similarly, $GLB\{a, b\} =$ the greatest +ve integer s.t. $d|a$ & $d|b$

$\therefore GLB\{a,b\} = GCD\{a,b\}$.

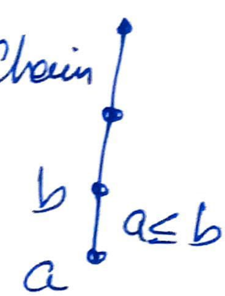


ex Set N of Natural nos. under divisibility forms a Lattice in which $a \vee b = lcm(a,b)$ & $a \wedge b = gcd(a,b)$.

ex $A = \{2,3,4,6\}$, divisibility relation of $|$.
Then $(A, |)$ is a poset but NOT a Lattice.
$\because LUB\{4,6\}$ does not exist in A.
$LCM(4,6) = 12 \notin A$

EX. Prove: Every chain is a Lattice.

Chain



Sol. Let $(L, \leq)$: Chain
Let $a, b \in L$. Since L is a Chain
$\Rightarrow a \leq b$ or $b \leq a$

W/o loss of generality, assume $a \leq b$
Then $a \vee b = b$ and $a \wedge b = a$

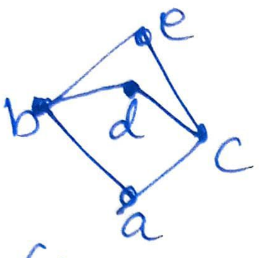
[Use always the Method of 1st vertex reached by upward or downward paths — LUB & GLB resp].

$\Rightarrow$ both $a \vee b$ & $a \wedge b \in L$

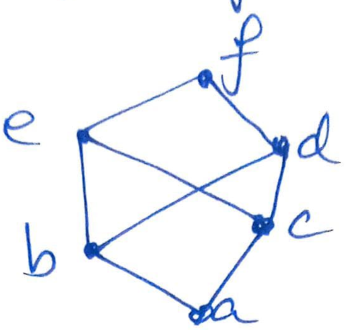
$\therefore L$ is chain.

Lat 3

EX Explain why the posets are not lattices.



(a)

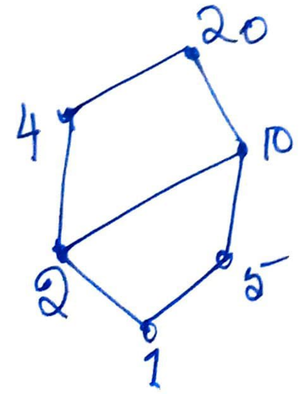


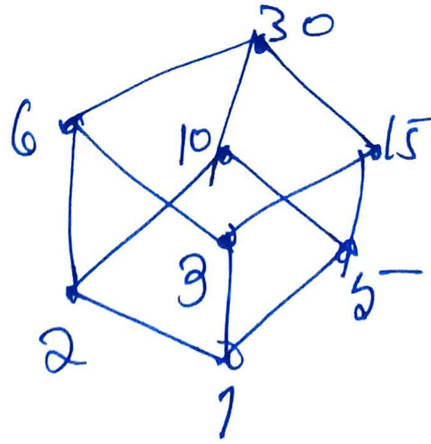
(b)

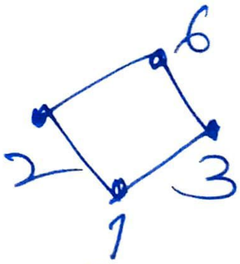
Sol. In (a), $d \vee e$ does not exist. (not unique)
$b \vee c$ does not exist (not unique)

In (b) neither $e \wedge d$ nor $b \vee c$ exist.
Note: $e \wedge d$ is not b ∵ c is also a lower bound of $\{e, d\}$ & $c \leq b$ is not true

ex Some more examples of lattices are:
Let D_n: Set of all positive divisors of n
∴ $D_{20} = \{1, 2, 4, 5, 10, 20\}$, $D_{30} = \{1, 2, 3, 5, 6, 10, 15, 30\}$







$D_6 = \{1, 2, 3, 6\}$

Lat (4)

Thm If $(L, \le)$ is a lattice then for any $a, b, c \in L$, the following results hold: (Properties of lattice)

1. $a \wedge a = a$, $a \vee a = a$ (Idempotent)
2. $a \wedge b = b \wedge a$, $a \vee b = b \vee a$ (Commutative)
3. $(a \wedge b) \wedge c = a \wedge (b \wedge c)$, $(a \vee b) \vee c = a \vee (b \vee c)$ (Associative)
4. $a \wedge (a \vee b) = a$, $a \vee (a \wedge b) = a$ (Absorption)

Pf:

1. $a \wedge a = GLB\{a, a\} = a \quad \therefore a \wedge a = a$

Similarly $a \vee a = LUB\{a, a\} = a \quad \therefore a \vee a = a$.

2. $a \wedge b = \inf\{a, b\} = \inf\{b, a\} = b \wedge a$

$a \vee b = \sup\{a, b\} = \sup\{b, a\} = b \vee a$

3. Let $b \wedge c = d$ then $d = GLB\{b, c\}$ (defn)

$\Rightarrow d \le b$, ~~$d \le a$~~ $d \le c$

Let $a \wedge d = e$ then $e = GLB\{a, d\}$

$\Rightarrow e \le a$ and $e \le d$

$\Rightarrow e \le a$ and $e \le b$ ~~and~~ $d \le c$

$[\because \le$ is Transitive & $d \le b, d \le c]$

3. Let $a \vee (b \vee c) = x$ and $(a \vee b) \vee c = y$

x is the join of a and $b \vee c$ ie, x is the LUB of a and $b \vee c$.

$\Rightarrow a \leq x$ and $b \vee c \leq x$

Now $b \vee c \leq x$

$\Rightarrow b \leq x$ and $c \leq x$ (defn of LUB

Hence the join of a and b is the LUB of a & b, from $a \leq x$ and $b \leq x$, we get

$a \vee b \leq x$

Further, $a \vee b \leq x$ and $c \leq x$

$\Rightarrow (a \vee b) \vee c \leq x \Rightarrow y \leq x$

Similarly, $x \leq y \therefore x = y$ (The equality in Lattices is from Antisymmetry property)

ie, $a \vee (b \vee c) = (a \vee b) \vee c$

&. By Principle of Duality,

$(a \wedge b) \wedge c = a \wedge (b \wedge c)$

$\therefore$ Both join & meet are Associative.

$a \leq a \vee (a \wedge b)$ — (1)

$\because$ $a \vee (a \wedge b)$ is the join of a and $a \wedge b$

Also $a \leq a$ and $a \wedge b \leq a$ — (*) (defⁿ of meet) GLB

Reflexive

$\therefore$ $a \vee (a \wedge b) \leq a \vee a$ [① Defⁿ of LUB & also ② Operate on both sides by $\vee$ with a ③ using (*)]

i.e, $a \vee (a \wedge b) \leq a$ — (2) (Idempotent)

$\therefore$ $a \vee (a \wedge b) = a$ (Antisymmetry in (1) & (2)).

Other result follows by Principle of Duality.

Thm For any a, b, c, d in a lattice $(L, \leq)$, the following properties hold:

(a) $a \leq b$ and $c \leq d \Rightarrow a \vee c \leq b \vee d$

(b) $a \leq b$ and $c \leq d \Rightarrow a \wedge c \leq b \wedge d$

Pf:- (a) Suppose $a \leq b$ and $c \leq d$

By defⁿ of join $\vee$,

$b \leq b \vee d$ and $d \leq b \vee d$

Corollary

Now $a \leq b$ and $b \leq b \vee d \Rightarrow$

$a \leq b \vee d$ [Transitive]

Similarly $c \leq d$ and $d \leq b \vee d \Rightarrow c \leq b \vee d$

$\therefore b \leq d$ is an upper bound of a and c.

Since $a \vee c$ is the LUB of a and c, we have $a \vee c \leq b \vee d$.

(b) Suppose: $a \leq b$ and $c \leq d$.

By defn of meet $\wedge$,

$a \wedge c \leq a$ and ~~$a \wedge c$~~ $a \wedge c \leq c$

By Transitivity

$a \wedge c \leq a$ and $a \leq b \Rightarrow a \wedge c \leq b$

and $a \wedge c \leq c$ and $c \leq d \Rightarrow a \wedge c \leq d$

$\Rightarrow a \wedge c$ is a lower bound of b and d.

$\therefore a \wedge c \leq b \wedge d$ since $b \wedge d$ is the GLB of b & d.

LHP

Thm Let $(L, \leq)$ be a lattice with least elt 0 & greatest elt 1, then for any $a \in L$

(A) $a \vee 1 = 1$ and $a \wedge 1 = a$

(B) $a \vee 0 = a$ and $a \wedge 0 = 0$

Pf. (A) Let $a \in L$, 1 is Greatest elt of L $\Rightarrow$

$a \vee 1 \leq 1$ —(1) (def^n of LUB)

Since $a \vee 1$ is $\sup\{a, 1\}$ $\Rightarrow$

$1 \leq a \vee 1$ —(2)

$\therefore$ for (1) & (2), $a \vee 1 = 1$ (Antisymmetry)

Also, $a \wedge 1$ is inf $\{a, 1\}$, $\therefore$

$a \wedge 1 \leq a$ —(3)

Also since $a \leq a$ & $a \leq 1$, [By Thm $a \leq b$ and $a \leq c \Rightarrow a \leq b \wedge c$]

$a \leq a \wedge 1$ —(4)

From (3) & (4), $a \wedge 1 = a$

(B) Let $a \in L$, 0: Least elt of L Reflexive

$\therefore 0 \leq a$ (0 is least elt.) and $a \leq a \Rightarrow \underline{a \vee 0 \leq a}$ LUB (LUB of LHS will be $\leq$ RHS)

$\therefore$ a is upper bound of 0 and a while $a \vee 0$ is the LUB of $\{0, a\}$.

Also, from def^n of $\vee$, $\Rightarrow a \leq a \vee 0$

Now, $a \vee 0 \leq a$ and $a \leq a \vee 0 \Rightarrow a \vee 0 = a$ (Antisymmetry)

Again $a \wedge 0$ is inf of a and 0, $\therefore a \wedge 0 \leq 0$

Since $0 \leq a$ and $0 \leq 0$, we have

$a \wedge 0 \leq 0$ and $0 \leq a \wedge 0 \Rightarrow a \wedge 0 = 0$

__Corollary__: 0 is the identity of $\vee$ and 1 is identity of $\wedge$ in a Bounded Lattice