Partial Order - A relation R on a set A is called a Partial Order if R is Reflexive, Antisymmetric & Transitive. The set together with the partial order R is called a Partially Ordered Set or a POSET / Poset denoted by (A, R) or simply A.

Ex. Show that Relation of Divisibility $(aRb \Leftrightarrow a|b)$ is a Partial order on Z^+ (set of positive integers).

Sol- Reflexive $x|x \quad \forall x \in Z^+ \therefore$ R is Reflexive

Antisymmetric If $x, y \in R$ & we have both xRy & yRx then

$xRy \Rightarrow y = ax$ for some $a \in Z^+$

$yRx \Rightarrow x = by$ for some $b \in Z^+$

$\Rightarrow y = ax = a(by) = (ab)y$

Since $y \neq 0 \Rightarrow ab = 1$

$a, b \in Z^+, ab = 1 \Rightarrow a = b = 1 \Rightarrow x = y$

$\therefore$ R is Antisymmetric

Transitive, let $a|b$, & $b|c$

$\Rightarrow a = mb,\ b = nc$ for some $m, n \in Z^+$

$a = m(nc) = pc$ where $p = mn \in Z^+$

$\Rightarrow a|c$ ∴ R is Transitive

∴ R is a Partial order on Z^+.

ex. Let A be a collection of subsets of a set S the relation set inclusion (subset of) $\subseteq$ is a partial order on A so $(A, \subseteq)$ is a Poset.

ex. Relation '$<$' on Z^+ is NOT a Partial order since $x \nless x\ \forall x \in Z^+$ – '$<$' is Not Reflexive

Note:- '$\preceq$' is used as symbol for partial order implying that '$\preceq$' can be any operation like '$|$' etc as long as it is Partial order.

Definitions:-

If R is a partial order on set A & R^{-1} PO(3)
(inverse relation) of R then R^{-1} is also partial order. The poset (A, R^{-1}) is the called the Dual of the poset (A, R) & the partial order R^{-1} is called the dual of the partial order R.

whenever $(A, \leq)$ is a poset then $(A, \geq)$ will be its Dual Poset.

The elements $a, b \in A$ are said to be Comparable if $a \leq b$ or $b \leq a$.

Every pair of elements need not be comparable.

ex. $aRb \Leftrightarrow a|b$, $2 \nmid 7$ and $7 \nmid 2$.

The word "partial" in Poset $\Rightarrow$ some elements may not be comparable.

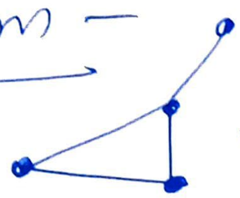
If every pair of elements in a poset A is comparable then it is said that A is <u>Linearly Ordered Set</u>. & the partial order is called a <u>Linear Order</u> & A is called a <u>Chain</u>.

ex. $(Z^+, \leq)$, $(Z^+, \geq)$ are chains

usual meaning of less than or equal to & greater than or equal to respectively

Hasse Diagram –

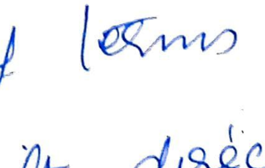
~~Dig~~ Graph:



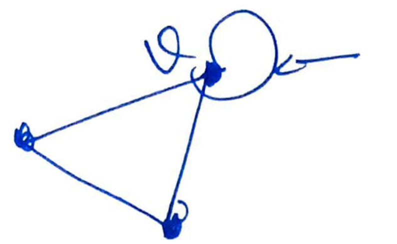
The points are called <u>vertices</u> (singular: vertex) & lines are called <u>edges</u>

Note: Graph Theory is a major & very popular & useful subject of Discrete mathematics in detail in the IVth Semester. For now, I will briefly explain the necessary terms only.

Digraph: – Graph with direction.



(Directed Graph)

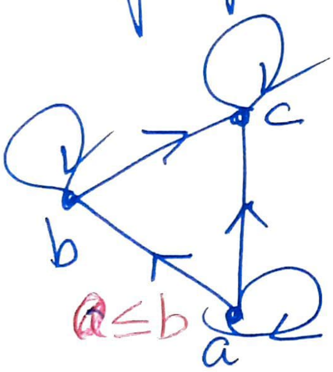


← loop or cycle of length 1
(1 ∵ 1 edge from vertex v to v)

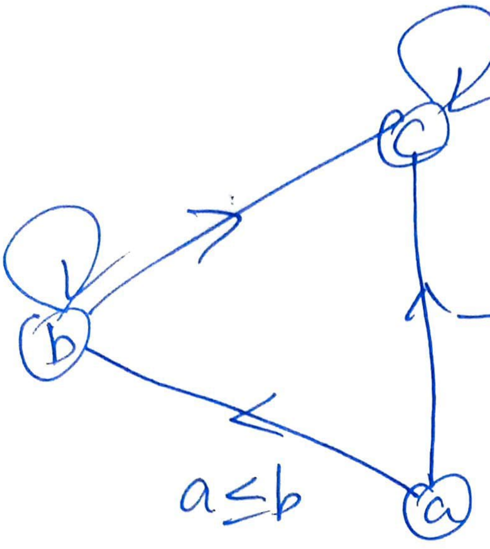
How to make Hasse Diagram from a Digraph

Note: A Relation can be represented using a digraph. A loop $v \xrightarrow{to} v \Rightarrow$ Reflexive property as in $c \to c$, $a \to a$ & $b \to b$ in following digraph.

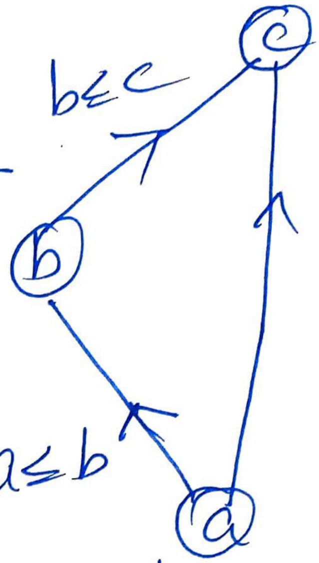
$a \leq b$ (a to b as given by the ~~dir~~ direction / arrow)



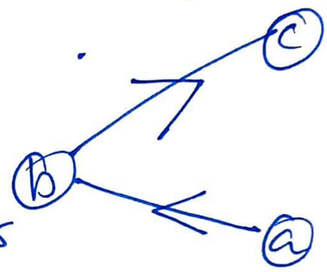
Steps –
1. Delete all loops (cycles of length 1)
2. Eliminate all edges implied by the Transitive Property.
ie, if $a \leq b$, $b \leq c \Rightarrow a \leq c$, Delete the edge a to c.
3. Omit the arrows. (∵ All the edges point upwards). Result is Hasse Diagram.



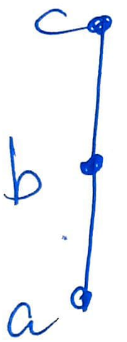
delete loops (Reflexive property) →



delete the Transitive edge a to c



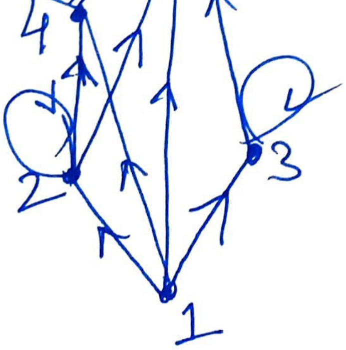
← Omit-arrows



(Hasse Diagram). Actually this Hasse Diagram is a <u>Chain</u>.

<u>Homework Exercise</u>

<u>Ex.</u> Let $A = \{1, 2, 3, 4, 12\}$. Consider the partial order of divisibility on A i.e, if $a, b \in A$, $a \leq b \iff a/b$. Draw the Hasse Diagram of poset $(A, \leq)$.

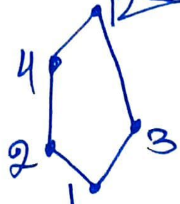


<u>Note</u>: The edge here represents the partial order of "divides".

<u>ex.</u>

this edge means 3 divides 12 or $3/12$.

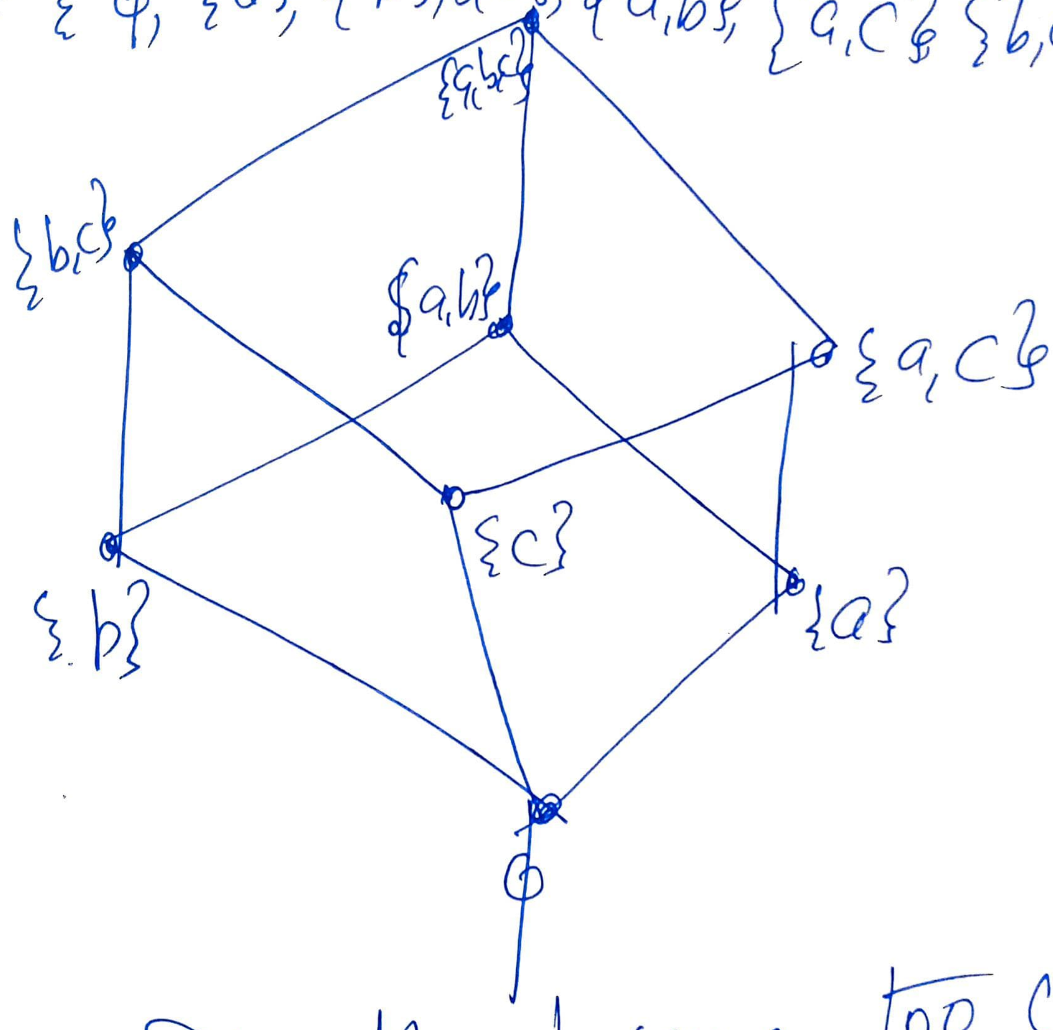
<u>Ans of this Ex.</u>



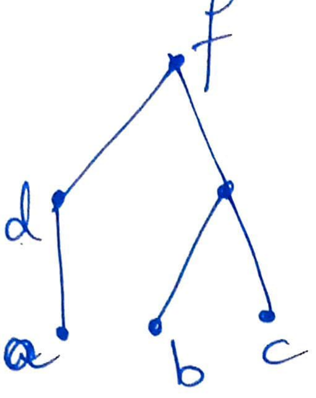
ex. $S = \{a, b, c\}$, $A = P(S)$ ← Power set

The Hasse Diagram of poset A with the partial order $\subseteq$ (set inclusion)

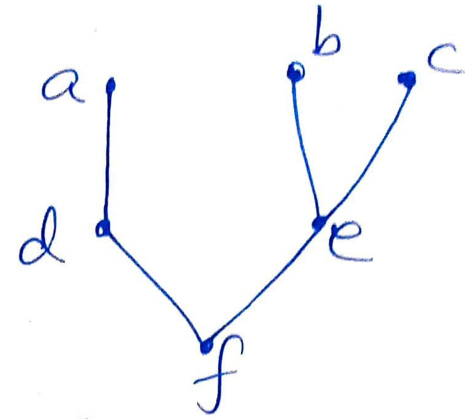
$P(S) = \{\phi, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\}\}$



Note: Digraphs become too complex to draw & understand so Hasse Diagrams are used.



$(A, \leq)$



Dual Poset $(A, \geq)$

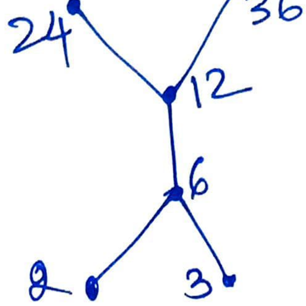
Definitions -

Minimal Element - Let $(P, \leq)$ be a Poset. An element 'a' in P is called a Minimal Element if there is no other element 'b' in P s.t. $b \neq a$ & $b \leq a$ i.e., if $b < a$ for no 'b' in P.

A Minimal Element in a poset need not be unique. Every finite Poset has at least one minimal element.

In Hasse Diagram, all those elements which appear at the lowest level of a Poset are Minimal elements.

ex. Poset $\{2, 3, 6, 12, 24, 36\}$ with divisibility relation, $a|b \longrightarrow$



Minimal Elements.

Def (2)

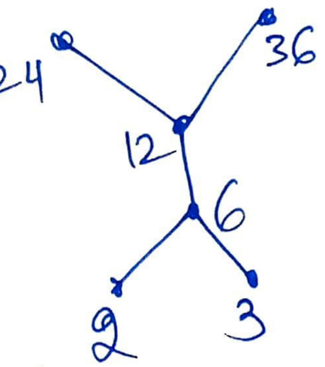
ex. Poset of Set of integers with usual '≤' (less than or equal to) — No Minimal Element.

Maximal Element — An element 'a' in Poset $(P, \leq)$ is called a maximal element if there is no 'b' in P s.t. $a \neq b$ & $a \leq b$ or if $a < b$ for no b in P

A Maximal elt. need not be unique. All those elts which appear at the highest levels of a Hasse Diagram are Maximal Elements.

A finite poset has at least 1 maximal elt.

ex.



Maximal Elts: 24, 36

ex. Poset Z (of integers) with usual "≤" — No maximal elt. & No Minimal elt.

ex. ~~Poset Z~~ Poset of non-negative Real Nos. with usual partial order "≤". Then

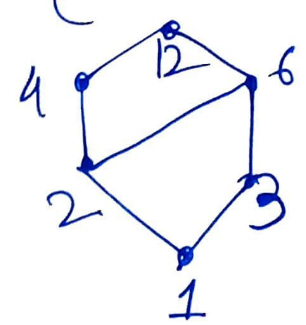
Minimal elt.: 0

No maximal elt.

Def 2.7

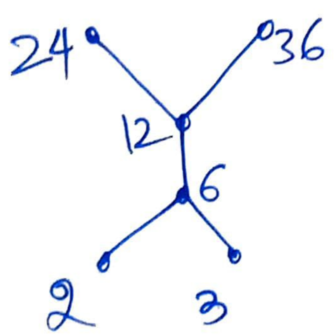
Least Element - $(P, \leq)$: Poset. If $\exists$ an element $a \in P$ s.t. $a \leq x$ $\forall x \in P$, then a is called the Least elt in P. This is also calling the First Element or Zero Element of P denoted by 0. Def^n $\Rightarrow$ if Least elt. exists, it is unique. It may not exist also.

ex. $X = \{1, 2, 3, 4, 6, 12\}$, $\leq$ defined as follows
$x \leq y \Longleftrightarrow x | y$ (x divides y)
$(X, \leq)$ is a Poset.



least elt : 1

ex.

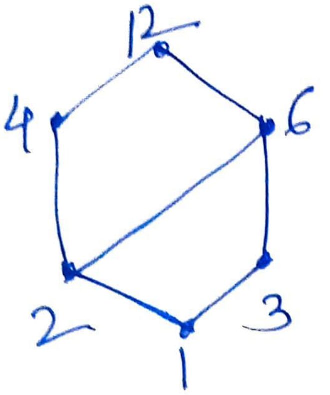


2 Minimal elts but NO Least elt.

Greatest Element - Also called Last elt. or Unit Element & denoted by 1. It may not exist, if it does then it is unique.

Def (3)

ex.



Greatest elt: 12

ex. $(P(S), \subseteq)$, $P(S)$: Power set of S.

Greatest elt: S

Least elt: ϕ

Upper Bounds & Least Upper Bound (LUB)

$(P, \leq)$: Poset, A: a subset of P. An elt. $x \in P$ is called an upper bound of A if $a \leq x \ \forall a \in A$.

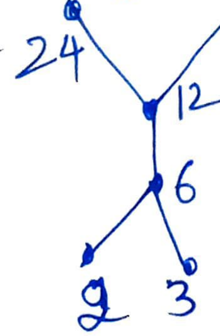
$P(S) = \{\phi, \{a\}, \{b\}, \{c\}, \{a,b\}, \{b,c\}, \{a,c\}, \{a,b,c\}\}$

ex. $X = \{a, b, c\}$, $(P(S), \subseteq)$ is Poset.

Let subset $A = \{\phi, \{a,b\}, \{b\}, \{c\}\}$

Then $\{a, b, c\}$ is an upper bound of A $\because$ every elt. of A is contained in $\{1,2,3\}$

ex.



, $\leq$ is $|$ (divides) or divisibility

$X = \{2, 3, 6, 12, 24, 36\}$,

Subset $A = \{2, 3, 6\}$

Upper bounds: 6, 12, 24, 36

$\because$ every elt. of A divides these elts.

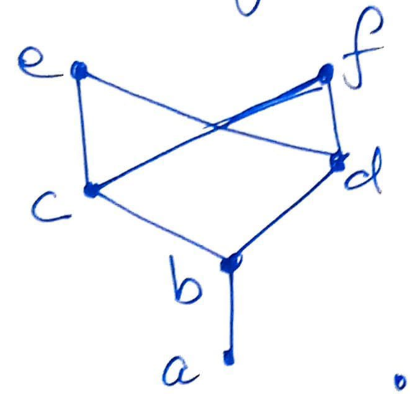
LUB or Supremum of A

Def^n (4)

where A is a subset of P & $(P, \leq)$: Poset.
An elt. $x \in P$ is said to be LUB or Supremum of A if x is an upper bound of A & $x \leq y$ for all upper bounds of A.

LUB - if it exists is unique & denoted by "lub", "LUB" or "sup". Sup of A is denoted by sup(A).

ex.

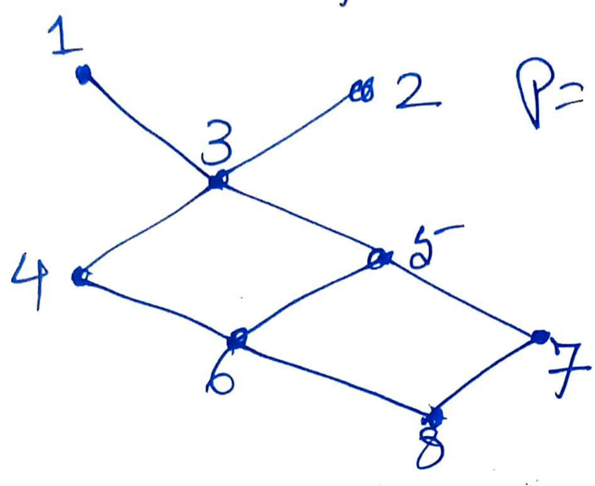


Let $A = \{b, c, d\}$,
Upper Bounds of A : e, f
Sup(A) does not exist
$\because$ upper bounds of A are Non-comparable.

Method: To find LUB(A), travel upward in the Hasse Diagram by upward paths (going up) from all the vertices of A. The 1st such vertex reached is the LUB(A). If such a vertex is reached then it has to be unique.

In given ex., if travelling upwards from b, c & d, (all the 3) the 1st vertices reached are e & f (NOT unique) so NO LUB(A)

ex



$P = \{1, 2, 3 - - - - 8\}$, $(P, \leq)$ is poset.
Let $A = \{4, 5, 7\}$.
Upper Bounds of A:
1, 2, 3
$LUB(A) = 3$

ex. P: Set of Natural Nos., $\leq$ be a Relation on P s.t. $a \leq b \Leftrightarrow a \mid b$, $(P, \leq)$: Poset
Let $A = \{a, b\}$; a, b - any 2 Natural nos
Then $LUB(A) = LCM(a, b)$
Least common multiple.

ex. if $A = \{2, 5\}$ then $LCM(2, 5) = 10$
2 & 5 can divide higher nos also but 10 is the LEAST of all such higher nos. (upper bounds) hence 10 is the LUB (2, 5)

Lower Bounds & Greatest Lower Bound

Let $(P, \leq)$: Poset, $A \subseteq P$. An elt. $x \in P$ is a Lower Bound or infimum of A if $x \leq a \; \forall a \in A$.

Def^n ⑥

$(P, \leq)$: Poset & $A \subseteq P$. An elt. $x \in P$ is a Greatest Lower Bound or infimum of A written as GLB(A) or inf(A) if x is a lower bound & $y \leq x$ for all lower bounds y of A. A GLB if it exists is unique.

Method in a Hasse Diagram —

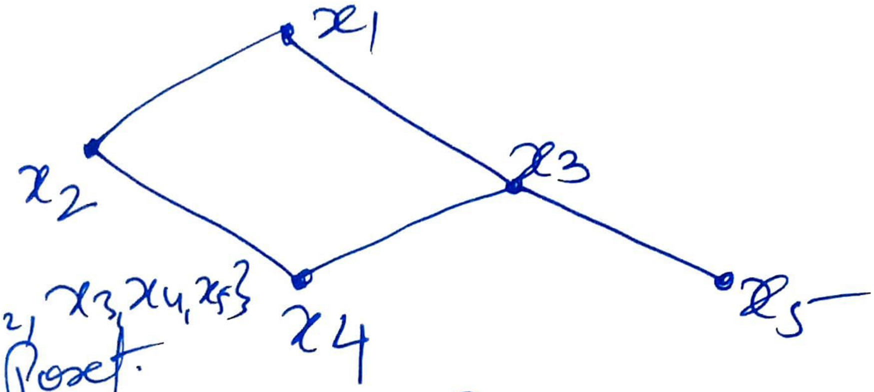
Say $A = \{a, b, c\}$. Travel downwards by all the downward paths from a, b & c. The 1st such vertex reached (if any) is the GLB(A).

ex. let $(N, \leq)$: Poset of Natural Nos.
$\leq$: usual meaning
let $A = \{5, 7, 9\}$.
Lower Bounds of A : 1, 2, 3, 4, 5
inf(A) = 5

GREATEST of all the lower bounds of A.

Def 17

ex



$P = \{x_1, x_2, x_3, x_4, x_5\}$
$(P, \leq)$: Poset.

$A = \{x_1, x_2, x_3\}$

Lower Bounds of A: x_4

$GLB(A)$: x_4

$B = \{x_3, x_4, x_5\}$ — No lower bounds exist.

$\therefore GLB(B)$ does not exist.

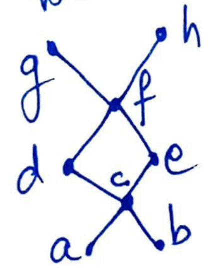
ex. P: Set of Natural nos, $\leq$ defined by $x \leq y \iff x \mid y$. $(P, \leq)$: Poset.

Let $A = \{a, b\}$: any subset of P with 2 Natural nos.

Then $GLB(A) = GCD(a, b)$ or $HCF\{a, b\}$
greatest-common divisor; highest-common factor

ex. if $A = \{6, 9\}$, $GCD(A) = 3$

Homework Exercise: Find if they exist, all upper bounds, all lower bounds, the LUB & GLB of $B = \{c, d, e\}$ in given poset.



Ans. $\{f, g, h\}$, $\{a, b, c\}$,
$LUB(B) = f$, $GLB(B) = c$