Thm Let $(L, \leq)$ be a lattice & $a, b, c \in L$.
Then the following implications hold:

(1) $a \leq b$ and $a \leq c \Rightarrow a \leq b \vee c$

(2) $a \leq b$ and $a \leq c \Rightarrow a \leq b \wedge c$

Pf. Def^n of Join $\Rightarrow$ $b \vee c$ is LUB $\{b, c\}$

(1) $\therefore b \leq b \vee c$

$\therefore a \leq b$ & $b \leq b \vee c \Rightarrow a \leq b \vee c$ (Transitive)

(2) Suppose $\underline{a} \leq b$ & $\underline{a} \leq c$ (lower bd on left)

$\Rightarrow$ $\underline{a}$ is a lower bound of $\{b, c\}$

$\Rightarrow a \leq b \wedge c$, the GLB $\{b, c\}$ **HP.**

Lattices as Algebraic Systems (Another def^n of Lattice):

A non empty set L together with 2 binary operations $\wedge$ (meet) & $\vee$ (join) is said to be a lattice if the following conditions are satisfied

1. Commutative properties: for any $a, b \in L$

$$a \wedge b = b \wedge a \quad \& \quad a \vee b = b \vee a$$

2. Associative properties: For any $a, b, c \in L$

$$a \wedge (b \wedge c) = (a \wedge b) \wedge c \quad \& \quad (a \vee b) \vee c = a \vee (b \vee c)$$

3. Absorption Properties: For any $a, b \in L$

$$a \wedge (a \vee b) = a \quad \& \quad a \vee (a \wedge b) = a$$

Direct Product of Lattices –

Thm Let $(L_1, \wedge, \vee)$ & $(L_2, \wedge, \vee)$ be 2 lattices & let $L = L_1 \times L_2$. Show that L is lattice under binary operation $\vee$ & $\wedge$ defined on L as

$$(a_1, b_1) \wedge (a_2, b_2) = (a_1 \wedge a_2,\ b_1 \wedge b_2) \ \&$$

$$(a_1, b_1) \vee (a_2, b_2) = (a_1 \vee a_2,\ b_1 \vee b_2)$$

$$\forall\ (a_1, b_1), (a_2, b_2) \in L$$

The lattice L is called the Direct Product of lattices $(L_1, \wedge, \vee)$ & $(L_2, \wedge, \vee)$.

Pf. Let (a_1, b_1) & (a_2, b_2) be any 2 elements of L.

L_1 is a lattice $\Rightarrow a_1 \wedge a_2$ & $a_1 \vee a_2 \in L_1$

Similarly $b_1 \wedge b_2$ & $b_1 \vee b_2 \in L_2$

$\therefore$ $(a_1 \wedge a_2, b_1 \wedge b_2) \in L$ & $(a_1 \vee a_2, b_1 \vee b_2) \in L$

$\Rightarrow$ both $\vee$ & $\wedge$ defined on L are binary operations

1. Commutative properties – Let $(a_1, b_1), (a_2, b_2) \in L$

Then $(a_1, b_1) \wedge (a_2, b_2) = (a_1 \wedge a_2,\ b_1 \wedge b_2)$

$= (a_2 \wedge a_1,\ b_2 \wedge b_1)$ ($\because L_1, L_2$ are lattices)

$= (a_2, b_2) \wedge (a_1, b_1)$

Similarly, $(a_1, b_1) \vee (a_2, b_2) = (a_2, b_2) \vee (a_1, b_1)$

2. Associative properties - Lat (10)

Let $(a_1,b_1),(a_2,b_2),(a_3,b_3)\in L$.

$$(a_1,b_1)\wedge[(a_2,b_2)\wedge(a_3,b_3)] = (a_1,b_1)\wedge(a_2\wedge a_3, b_2\wedge b_3)$$

$$= \cancel{(a_1\wedge(a_2\wedge}$$

$$= [a_1\wedge(a_2,a_3),\ b_1\wedge(b_2,b_3)]$$

$$= [(a_1\wedge a_2)\wedge a_3,\ (b_1\wedge b_2)\wedge b_3]\ (\because L_1, L_2 : \text{lattices})$$

$$= (a_1\wedge a_2, b_1\wedge b_2)\wedge(a_3,b_3)$$

$$= [(a_1,b_1)\wedge(a_2,b_2)]\wedge(a_3,b_3)$$

$\Rightarrow \wedge$ is Asso.

Similarly, $\vee$ is Asso.

3. Absorption properties - $(a_1,b_1),(a_2,b_2)\in L$

$$(a_1,b_1)\wedge[(a_1,b_1)\vee(a_2,b_2)] = (a_1,b_1)\wedge(a_1\vee a_2, b_1\vee b_2)$$

$$= (a_1\wedge(a_1\vee a_2),\ b_1\wedge(b_1\vee b_2)) = (a_1,b_1)$$

$(\because L_1, L_2 :$ lattices$)$

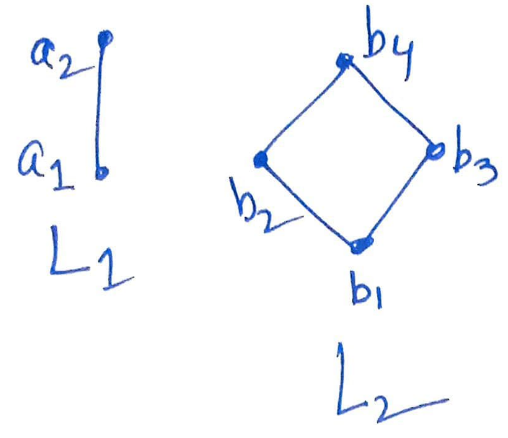
Similarly, $(a_1,b_1)\vee[(a_1,b_1)\wedge(a_2,b_2)]$

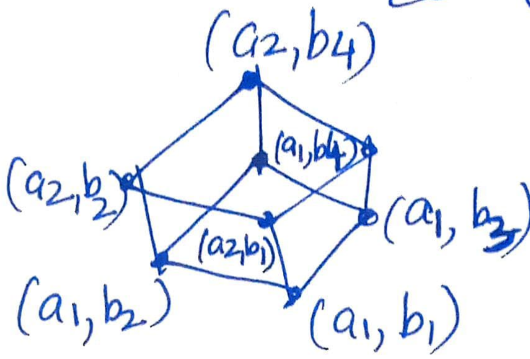
$$=(a_1,b_1)$$

$\therefore$ L is lattice.

HP.

Ex-





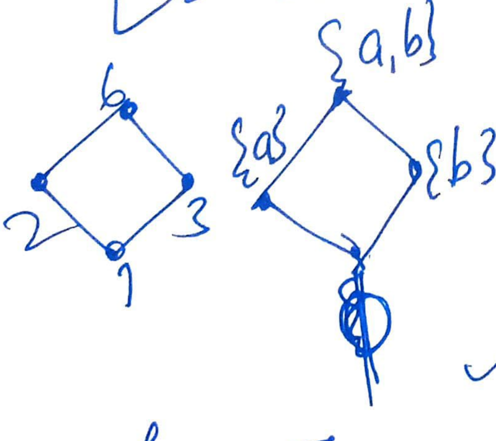
$L = L_1 \times L_2$

(all possible ordered pairs) i.e.
$2 \times 4 = 8$

Isomorphic Lattices – L_1, L_2: 2 lattices

$L_1 \cong L_2$ if $\exists$ a 1-1 mapping f from L_1 onto L_2 s.t. $f(a \wedge b) = f(a) \wedge f(b)$ &
$f(a \vee b) = f(a) \vee f(b)$ $\forall a, b \in L_1$

ex. lattice $D_6 = L = \{1, 2, 3, 6\}$ under divisibility & the lattice $(P(S), \subseteq)$ where $S = \{a, b\}$. Then $L \cong P(S)$. Define a mapping $f : L \to P(S)$



$f(1) = \phi$
$f(2) = \{a\}$
$f(3) = \{b\}$
$f(6) = \{a, b\}$

f is 1-1 & onto

Also $f(a \wedge b) = f(a) \wedge f(b)$
$f(a \vee b) = f(a) \vee f(b)$ $\forall a, b \in L$

$\therefore f$ is an Iso. & $(L, |) \cong (P(S), \subseteq)$.

Bounded Lattice – The Least & Greatest elements of a lattice, if they exist are also called universal lower bound & universal upper bound resp. Least elt is denoted by 0 & Greatest elt. by 1.

A lattice L is said to be **Bounded lattice** if L contains both Greatest & Least elts 0 & 1. Every finite lattice is a Bounded Lattice.

ex. $L = \{1, 2, 3, 6\} = D_6$, $\leq : '|'$ (divisibility)

$(L, \leq)$ is Bounded Lattice. Least elt. is 1 & Greatest elt. = 6.

ex. Lattice $(N, \leq)$, N: Natural nos, $\leq$ is ~~usual~~ usual "less than or equal to" relation on N, is NOT Bounded ∵ NO Greatest elt., Least elt = 1.

Complete Lattice – Lattice is Complete Lattice if every non empty subset of L has its LUB & GLB in L.

Rm. If L is Complete lattice then it is a Bounded Lattice.

Pf Let L: Complete lattice. Then every non empty subset of L has LUB & GLB. In particular, L itself has LUB & GLB.

∴ L is a Bounded lattice.

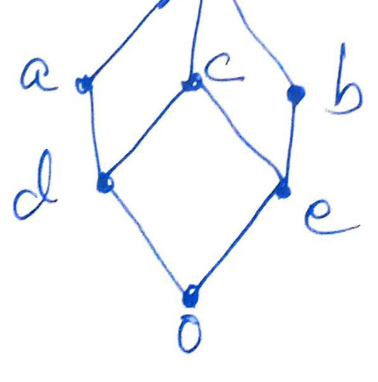
Complement of an elt. —

Let L: Bounded lattice with universal lower & upper bounds 0 & 1. For any element $a \in L$, an element $a' \in L$ is said to be a Complement of a if

$a \vee a' = 1$ and $a \wedge a' = 0$

⇒ if a is a complement of a' then a' is also a complement of a.

An elt. may have more than 1 complement.

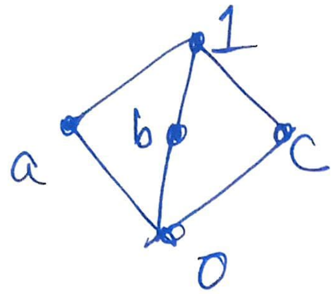


b & e are complements of a
c has no complement
1 is complement of 0 & 0 is complement of 1.

Complemented Lattice – A Bounded Complemented Lattice L is Complemented Lattice if every element in L have a complement.

ex: $(P(S), \subseteq)$ is Complemented Lattice

e.x.



both a & b are complements of c.

Complemented Lattice

Thm: The Dual of a Complemented Lattice is a Complemented Lattice.

Pf: Let $(A, \leq)$: Complemented Lattice. Its dual is also a lattice with 0 & 1.

Now $a \leq 1 \ \forall a \in A \Rightarrow 1 \geq a \ \forall a \in A$

$\Rightarrow$ 1 is the universal lower bound of $(A, \geq)$

Similarly, 0 is the universal upper bound of $(A, \geq)$

Let any $a \in A$

Since $(A, \leq)$ is a Complemented lattice,

$\exists \ \bar{a} \in A$ s.t. $a \vee \bar{a} = 1$ and $a \wedge \bar{a} = 0$ ($\bar{a}$ is complement of a)

$\therefore$ 0 is the GLB $\{a, \bar{a}\}$ in $(A, \leq)$

Comp (3)

1 is the LUB $\{a, \bar{a}\}$ in $(A, \leq)$

This $\Rightarrow$ $0 \leq a,\ 0 \leq \bar{a}$ & $a \leq 1,\ \bar{a} \leq 1$

$\Rightarrow a \geq 0,\ \bar{a} \geq 0$ & $1 \geq a,\ 1 \geq \bar{a}$

$\Rightarrow$ 0 & 1 are upper & lower bounds of a & $\bar{a}$ in $(A, \geq)$

Let x be any other upper bound of a & $\bar{a}$ in $(A, \geq)$ so, $a \geq x$, & $\bar{a} \geq x$

$\Rightarrow x \leq a$ & $x \leq \bar{a}$

$\Rightarrow x \leq 0$ Since 0 is the GLB $\{a, \bar{a}\}$ in $(A, \leq)$

$\Rightarrow 0 \geq x$

$\Rightarrow$ 0 is LUB $\{a, \bar{a}\}$ in $(A, \geq)$

$\therefore$ in $(A, \geq)$, $a \vee \bar{a} = 0$

Similarly, $a \wedge \bar{a} = 1$ in $(A, \geq)$

$\Rightarrow$ complement of a is $\bar{a}$ in $(A, \geq)$

$\therefore (A, \geq)$ is a complemented lattice.