Thm If lattices L & M are Isomorphic & f denotes an isomorphism then f preserves the ordering relation. i.e., for any $a, b \in L$, $a \leq b \Rightarrow f(a) \leq f(b)$

Pf Let $f : L \to M$ be an Iso.

Let $a, b \in L$ s.t. $a \leq b$

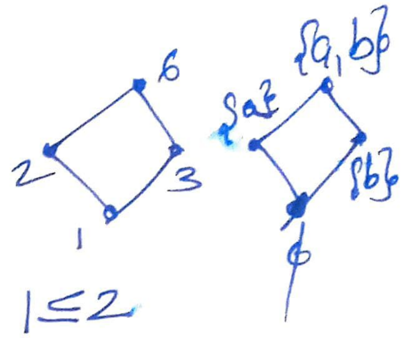
Then $a = a \wedge b$

$\Rightarrow f(a) = f(a \wedge b)$

$= f(a) \wedge f(b)$ ($\because f$ is Iso.)

$\Rightarrow f(a) \leq f(b)$

$\therefore f$ preserves ordering relation



$1 \leq 2$

$1 = 1 \wedge 2$

$= \cancel{\{a\} \wedge \{b\}}$

$= \phi \wedge \{a\}$

$\phi \leq \{a\}$

where

$f(1) = \phi$

$f(2) = \{a\}$

$f(3) = \{b\}$

$f(6) = \{a,b\}$

Thm Let L, M: Lattices. If $f : L \to M$ is an Iso. of L onto M & 'a' is the Least elt. of L then $f(a)$ is the Least elt of M.

Pf Let a: Least elt. of L. Then $a \leq x\ \forall x \in L \Rightarrow f(a) \leq f(x)$ (def^n) $\forall f(x) \in M$

Since f is onto, any elt. $y \in M$ is of the form $f(x)$ for some $x \in L$. (above Thm) $\therefore f(a)$ is Least elt. of M

AP

Lat Iso (2)

Thm If $f: L \to M$ is an Iso. & $x, y \in M$ s.t. $x < y$ then $\exists\, a, b \in L$ s.t. $f(a) = x$, $f(b) = y$ and $a < b$.

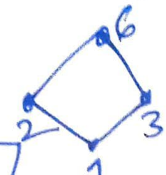
Pf Since f is onto & $x, y \in M$, $\therefore \exists\, a$ & c in L s.t. $f(a) = x$ and $f(c) = y$

Now, $f(a \vee c) = f(a) \vee f(c)$ ($\because f$ is Iso.)

$= x \vee y$

$= y$ — (1) ($\because x < y$) (def^n of LUB)

$x=1$
$y=2$



We know, $a \leq a \vee c$ (def^n of LUB)

If $a = a \vee c$ then

$f(a) = f(a) \vee f(c) = y$ — [from (1) & $f(a) = x$]

$\Rightarrow x = y$ — Contradiction

$\therefore a < a \vee c$

Now, we take $b = a \vee c$

Then $a < b$ and $f(a) = x$, $f(b) = y$

HP.

This Thm $\Rightarrow$ 2 lattices can be represented by the same diagram $\Leftrightarrow$ they are Isomorphic.

Comp(1)

Thm Every finite Lattice is Complete Lattice

Pf: Note: Complete Lattice L: every non empty Subset of L has its LUB & GLB in L.

Let $(L, \wedge, \vee)$ be any finite Lattice. Let S be any non empty Subset of L. Then S is also a finite Set. Let

$S = \{x_1, x_2, ---, x_m\}$, ~~Let~~

Then $x_1 \wedge x_2 \wedge x_3 \wedge \cdots \wedge x_m$ and $x_1 \vee x_2 \vee x_3 \cdots \vee x_m$

are GLB & LUB of S in L. $\therefore$ L is Complete.

Thm. If $(P, \leq)$ is a Poset with greatest elt. 1 s.t. every non empty Subset S of P has infimum, then $(P, \leq)$ is a Complete Lattice.

Pf. Let S: non empty Subset of P.

To prove: Sup S exists. (as per def^n of Complete Lattice)

Since 1 is the ~~Grea~~ Greatest elt. of P

$\therefore x \leq 1 \; \forall x \in P$ and (def^n of 1) $\Rightarrow s \leq 1 \; \forall s \in S$

$\Rightarrow$ 1 is an upper bound of S.
(1 is on RHS of $\leq$)

Let U : Set of all upper bounds of S Compl (2)

$\Rightarrow U$ is non empty $(\because 1 \in U)$

$\Rightarrow$ by given condition inf. U exists
[every nonempty subset has inf.]

Let $k = \inf U$

Now $s \in S \Rightarrow s \leq x \quad \forall x \in U$ (~~def^n of inf.~~) {U: set of all upper bds of S)

$\Rightarrow$ each elt. of S is a lower bound of U [$\because s$ is on LHS of $\leq$] $\Rightarrow s$ is lower bd.

$\Rightarrow s \leq k \quad \forall s \in S$ [$\because k$ is GLB or inf.]

$\Rightarrow k$ is an upper bd. of S [$\because k$ is on RHS of $\leq$]

But k is a lower bd of U [$\because k$ is GLB or inf. of U]

$\Rightarrow k \leq x \ \forall x \in U$ [def^n of lower bd]

$\Rightarrow k \leq x$ for all upper bds of S (def^n of U)

$\Rightarrow k = \operatorname{Sup.} S$ [def^n of sup or LUB]

$\therefore P$ is Poset in which every nonempty subset has sup & inf & $\therefore P$ is Complete Lattice

H.P.

Thm. If $(P, \leq)$ is a Poset with least elt. 0 s.t. every non empty subset of P has sup then P is a complete lattice. Conf ②

Pf. Proof follows by Principle of Duality from above Thm.

HW Exercise — Do this proof without the Principle of Duality.

Thm In a lattice L with universal lower bd 0 & universal upper bd 1, Prove: 0 is the unique complement of 1 & 1 is the unique complement of 0.

Pf Clearly, $0 \wedge 1 = 0$ and $0 \vee 1 = 1$

$\Rightarrow$ 0 & 1 are complements of each other (def^n).

We prove: 1 is the only complement of 0.

Suppose: a is any other complement of 0, then

$0 \wedge a = 0$ and $0 \vee a = 1$ (def^n of complement)

But we know that $0 \vee a = a$

$\Rightarrow a = 1 \Rightarrow$ complement of 0 is unique.

Similarly, 0 is the only complement of 1.

Thm Dual of a lattice is a lattice. Comp (4)

Pf Let (L, R) be a lattice & $(L, \bar{R})$: its Dual.

$\bar{R}$ is defined as $x \bar{R} y \Leftrightarrow y R x$

Let $x, y \in L$. Since (L, R) is a lattice,

$\sup\{x, y\}$ exists in (L, R)

Let $x \vee y = \sup\{x, y\}$ in (L, R)

Then $x R (x \vee y)$ and $y R (x \vee y)$

$\Rightarrow (x \vee y) \bar{R} x$ and $(x \vee y) \bar{R} y$ (def^n dual)

$\Rightarrow x \vee y$ is a lower bound of (x, y) in $(L, \bar{R})$ (def^n of lower bd)

We show: $x \vee y$ is GLB $\{x, y\}$ in $(L, \bar{R})$

Let z: any lower bound of $\{x, y\}$ in $(L, \bar{R})$

Then $z \bar{R} x$ and $z \bar{R} y$ (def^n: lower bd)

$\Rightarrow x R z$ and ~~$z R y$~~ $y R z$ (def^n of upper bd)

$\Rightarrow z$ is an upper bound of $\{x, y\}$ in (L, R)

$\Rightarrow (x \vee y) R z$ ($\because x \vee y = \underline{\underline{\sup}}\{x, y\}$ in (L, R)) LUB

$\Rightarrow z \bar{R} (x \vee y)$

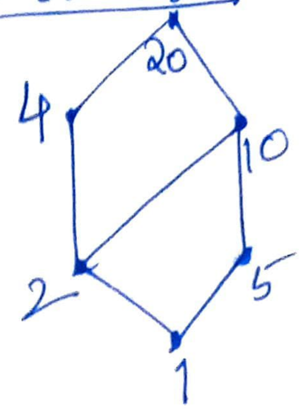
$\Rightarrow x \vee y$ is GLB $\{x, y\}$ in $(L, \bar{R})$

Similarly, $x \wedge y$ is LUB of $\{x, y\}$ in $(L, \bar{R})$ $\therefore (L, \bar{R})$ is lattice

Element appearing on RHS of R or $\leq$ is upper bd on LHS – lower bd

Homework Exercise: Contd.

In D_{20}



Verify that 2 & 10 have no complements.

& In D_{30}, every elt. has a complement.

EX. The lattice $(P(S), \subseteq)$, every elt. has a complement. If $A \in P(S)$ then $\bar{A}$ (the set complement) has the properties $A \vee \bar{A} = S$ & $A \wedge \bar{A} = \phi$ where $\vee$ is Union & $\wedge$ is intersection ∴ the set complement is also the complement in the lattice L.

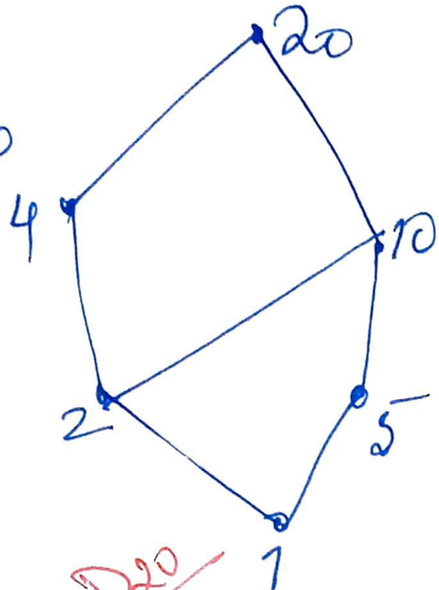
Distributive Lattice – If for any $a, b, c \in L$

$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$

& $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$ hold

then the lattice is Distr. Lattice.

D20



D20

2 and 10
Verify: ~~2~~ have no Complements.

$$10 \begin{cases} 2 \vee 10 = 10 \\ 2 \wedge 10 = 2 \end{cases}$$

$$5 \begin{cases} 2 \vee 5 = 10 \\ 2 \wedge 1 = 1 \end{cases}$$

$$1 \begin{cases} 2 \vee 1 = 2 \\ 2 \wedge 1 \quad 1 \end{cases}$$

$$4 \begin{cases} 2 \vee 4 = 4 \\ 2 \wedge 4 = 2 \end{cases}$$

$$20 \begin{cases} 2 \vee 20 = 20 \\ 2 \wedge 20 = 2 \end{cases}$$

Def^n of 2's Complement a':

$2 \vee a' = 1 \equiv 20$

and $2 \wedge a' = 0 \equiv 1$

Def.^{n} of Complement not satisfied by any element for 2 ∴ 2 has no Complements.

10 has no Complement – HW Exercise

HW Exercise – Verify that every elt. in D_{30} has a complement