Lipschutz P223

BA(1)

Boolean Algebra (BA) - Let B: non empty set with also 4 binary operations + and $*$, a unary operation ' (complement) and 2 distinct element 0 and 1. Then B is called a Boolean Algebra if the following axioms hold for any $a, b, c \in B$.

[B1] Commutative Laws: The operations + and $*$ are commutative.

$a+b=b+a$ (Some books use $\cdot$ in place of $*$)

$a*b=b*a \quad \forall a,b \in B.$

[B2] Identity Laws: For any $a \in B$

$a+0=a$ and $a*1=a$

0, 1: respective identities of + & $*$.

[B3] Distributive Laws:

For any $a,b,c \in B$,

$a+(b*c) = (a+b)*(a+c)$ Not allowed in ordinary algebra

& $a*(b+c) = (a*b)+(a*c)$

[B4] Complement Laws: For each $a \in B$, $\exists$ an elt a' in B s.t. $a+a'=1$ and $a*a'=0$

[B5] $\exists$ at least elt $a, b \in B$ s.t. $a \neq b$

BA ② defn

We denote BA by $\langle B, +, *, ', 0, 1 \rangle$

0 & 1: zero elt. (identity for $+$) and ~~one~~ unit elt.. (identity for $*$)

a': Complement of a in B

$a * b$ is usually written as ab

symbols $\vee$ & $\wedge$ also used for $+$ & $*$ resp

$\bar{a}$ also used for complement a'.

Boolean Pg P224

Ex. 5.1. S: non empty set, $P(S)$: Power set of S (B.A)

$\langle P(S), \cup, \cap, ', \phi, S \rangle$ is a Boolean Algebra

Sol. Here $+$ is $\cup$ (union)

$*$ is $\cap$ (intersection)

0 is ϕ

1 is S

1. Commutative Laws:

From set theory, we know

$A \cup B = B \cup A$ and $A \cap B = B \cap A$

$\forall A, B \in P(S)$

$\therefore$ Comm. Laws satisfied.

2. Identity Laws

we know ϕ & $S \in P(S)$ s.t.

$A \cup \phi = A$ and $A \cap S = A$ for any $A \in P(S)$

$\therefore \phi$ & S act as 0 & 1 resp.

3. Distributive Laws : From set theory,

we know, $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

and ~~A~~ $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

$\forall A, B, C \in P(S)$

$\therefore$ both operations $\cup$ & $\cap$ distribute over each other.

4. **Complement Laws** –

BA (B)

For any $A \in P(S)$, $S - A \in P(S)$ in $P(S)$

$\therefore$ Every elt $A \in P(S)$ has complement $A' = S - A \in P(S)$

$\therefore (P(S), \cup, \cap, ', \phi, S)$ is a BA

If S has n elts then $P(S)$ contains 2^n elts.

P 228

EX. 5.3 Show that the set $B = \{0, 1\}$ with the operation $+$, $*$ and $'$ defined by the foll. tables is a BA.

+	0	1
0	0	1
1	1	1

*	0	1
0	0	0
1	0	1

'	0	1
	1	0

Sol. It is clear from the table that $+$ & $*$ are binary op^{n}s on B & $'$ is a unary operation on B

1. **Comm. laws:** Since tables for $+$ & $*$ are symmetrical about main diagonals, both operations are commutative

2. Identity Laws : Clear from the BA ⓒ tables that $a+0=a$ & $a*1=a$ $\forall a \in B$

$\therefore$ 0 is the identity for + & 1 is identity for *.

3. It is easy to verify that both + & * are distributive over each other

ie, $a+(b*c)=(a+b)*(a+c)$

and $a*(b+c)=(a*b)+(a*c)$

$\forall a,b,c \in B$

4. Complement Laws

Given $0 \in B$, $\exists$ $1 \in B$ s.t. $0+1=1$ & $0*1=0$

& given $1 \in B$, we have $0 \in B$ s.t.

$1+0=1$ & $1*0=0$

$\therefore$ for every $a \in B$, $\exists$ $a' \in B$ s.t.

$a+a'=1$ & $a*a'=0$

$\therefore$ $(B,+,*,',0,1)$ is a BA.

De Morgan's Laws — Chauhan, P344 BA

In the Boolean Algebra B, $\forall\, a, b \in B$

(1) $(a+b)' = a'b'$

(2) $(ab)' = a' + b'$

Pf. (1) We show $(a+b) + (a'b') = 1$

and $(a+b)(a'b') = 0$

Now, $(a+b) + (a'b') = [(a+b) + a'][(a+b) + b']$

(Distr.)

$= [a' + (a+b)][(a+b) + b']$

(Comm ~~Asso~~)

$= [(a'+a) + b][a + (b+b')]$

(Asso)

$= (1+b)(a+1)$

(Boundedness)

$= 1 \cdot 1$ $\quad 1+b=1$

$= 1$ —(1) (Idempotent)

Asso,

BA (E)

$$(a+b)(a'b') = a(a'b') + b(a'b') \quad \text{- Distr}$$

$$= (aa')b' + b(b'a') \quad \text{- Comm + Asso}$$

$$= (aa')b' + (bb')a' \quad \text{- Asso}$$

$$= 0 \cdot b' + 0 \cdot a' \quad (\text{Boundedness } \cancel{\text{Complement}} \text{ Laws } a \cdot 0 = 0)$$

$$= 0 + 0 \quad (\text{idempotent})$$

$$= 0 \quad \text{—— (2)}$$

To prove: From (1) & (2), $(a+b)' = a'b'$

(2) To prove: $a' + b'$ is the complement of ab, we show

$$(ab) + (a'+b') = 1 \;\&\; (ab)(a'+b') = 0$$

Now, $(ab) + (a'+b') = (a'+b') + (ab)$

$$= [(a'+b') + a][(a'+b') + b] \quad (\text{Distr})$$

$$= [(a + (a'+b')][(a'+b') + b] \quad (\text{Comm + Asso})$$

$$= [(a + a') + b'][a' + (b' + b)] \text{- Asso.}$$

BA (F)

$= (1+b')(a'+1)$ (~~Boundedness~~ Boundedness)

$= 1 \cdot 1$ (Boundedness) ~~(Idempotent)~~

$= 1$ — (3) idempotent

Also, $(ab)(a'+b') = [(ab)a'] + [(ab)b']$

$= a'(ab) + (ab)b'$ — Distr

$= (aa')b + a(bb')$ — Asso

$= 0 \cdot b + a \cdot 0$

$= 0 + 0$ (Boundedness)

$= 0$ — (4) (Idempotent)

from (3) & (4), $(ab)' = a' + b'$.

Associative Laws of Boolean Algebra (BA) -

$a+(b+c) = (a+b)+c$

Pf. $a+(b+c) = [a+(b+c)]\cdot 1$

$= [a+(b+c)](c+c')$

$= [a+(b+c)]c + [a+(b+c)]c'$ - Distr

$= [ca + c(b+c)] + [c'a + c'(b+c)]$ - Distr

$= [ca + c] + [c'a + c'b + c'c]$ - Absorption + Distr

$= [c + ca] + [c'(a+b) + 0]$ - Distr + Comm

$= c + c'(a+b)$ - Absorption

$= (c+c')[c+(a+b)]$ - Distr

$= 1\cdot [c+(a+b)]$

$= c+(a+b)$

$= (a+b)+c$ - Comm

$\therefore$ $a+(b+c) = (a+b)+c$

other result follows by Principle of Duality.