One more defⁿ of Boolean Algebra (BA) : (BA) def (1)

A Boolean Algebra is a Distributive, Complemented Lattice having at least 2 distinct elements as well as a ~~0~~ Zero element 0 & a One element 1. The BA can be represented by the system

$$\mathbb{B} = \langle B, +, \cdot, ', 0, 1 \rangle$$

where B is a set, + & · are binary operations & ' is a unary operation (complementation) such that the following axioms hold:

1. There exist at least 2 elts $a, b \in B$ s.t. $a \neq b$.
2. $\forall a, b \in B$ (a) $a+b \in B$ (b) $a \cdot b \in B$
3. Commutative Laws $\forall a, b \in B$
   (a) $a+b = b+a$ (b) $a \cdot b = b \cdot a$
4. (a) $\exists\, 0 \in B$ s.t. $a+0 = a$, $\forall a \in B$ Existence of Zero
   (b) $\exists\, 1 \in B$ s.t. $a \cdot 1 = a$ $\forall a \in B$ Existence of Unit
5. $\forall a, b, c \in B$
   (a) $a+(b \cdot c) = (a+b) \cdot (a+c)$
   (b) $a \cdot (b+c) = (a \cdot b) + (a \cdot c)$
   ] Distributive Laws
6. Existence of Complement
   $\forall a \in B$, $\exists a' \in B$ s.t.
   (a) $a + a' = 1$ (b) $a \cdot a' = 0$, a': complement of a

BA
defn ②

$\vee$ & $\wedge$ can be used in place of $+$ & $\cdot$

We also write ab for $a \cdot b$.

In BA, sum, join & disjunction are interchangeable as are product, meet & conjunction

~~Thm~~ Thm

BA (7)

Thm 1 Idempotent Laws - In a Boolean Algebra B, $\forall a \in B$,

(1) $a+a=a$

(2) $aa=a$

Pf: (1) Since + is binary operation on B

$\therefore a \in B \Rightarrow a+a \in B$ (Closure)

$(a+a)=(a+a)\cdot 1$ - Axiom 4(b) - exis. 91

$=(a+a)(a+a')$ Axiom 6(a) - compl.

$= a+aa'$ - Axiom 5(a) - Distr

$= a+0$ - Axiom 6(b) - compl.

$= a$ - Axiom 4(a) exis. 0

$\therefore a+a=a$

(2) - Principle of Duality.

$aa=aa+0$ Axiom 4(a)

$=aa+aa'$ Axiom 6(b)

$=a(a+a')$ - Distr

$= a\cdot 1$ - Axiom 6(a)

$= a$ Axiom 4(b)

HP

Thm 2 Boundedness Laws - In a Boolean Algebra B, $\forall a \in B$

~~$a+0$~~ (1) $a+1=1$

(2) $a\cdot 0=0$

BA 9

Pf- (1) $\forall a \in B$,

$a+1 = (a+1)\cdot 1$ Axiom 4(b) - ~~exis~~ 1

$= (a+1)(a+a')$ - Axiom 6(a) - comp.

$= a+(1\cdot a')$ - Distr [Axiom 5(a)]

$= a+a'$ - Axioms 4(b) & 2(b) - Exist 1, Comm

$= 1$ - Axiom 6(a) - complt.

$\therefore a+1 = 1 \ \forall a \in B$

(2) by Principle of Duality

Absorption Laws - In a Boolean Algebra B, for every pair of elts a & $b \in B$

(1) $a+(ab) = a$

(2) $a(a+b) = a$

Pf-

$a+(ab) = (a\cdot 1)+(ab)$ Axiom 4(b) - exis. f 1

$= a(1+b)$ - Distr Axiom 5(b)

$= a(b+1)$ - Comm

$= a\cdot 1$

$= a$

$\therefore a+(ab) = a$

(2) - P. of Duality

Thm 3. The elements 0 & 1 are unique (BA Complem)

Pf. Suppose there are 2, zero elements 0_1 & 0_2. For each $a_1 \in B$ & $a_2 \in B$,

By Axiom 4(a) - Existence of zero

$$a_1 + 0_1 = a_1 \quad \& \quad a_2 + 0_2 = a_2$$

Let $a_1 = 0_2$ & $a_2 = 0_1$, then

$$0_2 + 0_1 = 0_2 \quad \& \quad 0_1 + 0_2 = 0_1$$

But by Axiom 2(a), $0_2 + 0_1 = 0_1 + 0_2$ (closure & comm.)

$\Rightarrow 0_2 = 0_1$

By Principle of Duality, elt. 1 is unique as well.

(BA-complement (2))

Thm $\forall a \in B$, there exists a unique complement $\bar{a}$.

Pf. Suppose a has 2 complements $\bar{a}_1$ & $\bar{a}_2$. Then by Axiom 6(a), 6(b) (exis. of complements)

$a + \bar{a}_1 = 1, \quad a + \bar{a}_2 = 1$

$a \cdot \bar{a}_1 = 0, \quad a \cdot \bar{a}_2 = 0$

Hence, $\bar{a}_1 = 1 \cdot \bar{a}_1$ Axiom 2(b) & 4(b) closure exis. of 1

$= (a + \bar{a}_2) \cdot \bar{a}_1$ Axiom 6(a) compl

$=$ ~~$a+$~~

$= a\bar{a}_1 + \bar{a}_2 \bar{a}_1$ Distr Axiom 5(b)

$= 0 + \bar{a}_2 \bar{a}_1$ Axiom 6(b) - compl

$= a\bar{a}_2 + \bar{a}_1 \bar{a}_2$ Axioms 6(b) & 2(b) compl closure

$= (a + \bar{a}_1) \cdot \bar{a}_2$ - Distr $=$ ~~$(a+\bar{a}_2) \cdot \bar{a}_2$ - Distr~~

$= 1 \cdot \bar{a}_2$ — Axiom 6(a) - compl.

$= \bar{a}_2$ Axiom 4(b) - exist-of 1

$\therefore \bar{a}_1 = \bar{a}_2 \therefore$ Complement is unique.

HP.

Thm 8.5 In every Boolean algebra, BA (h)

The 0 elt & the 1 elt of B are distinct &

(1) $0' = 1$

(2) $1' = 0$

Pf: Let $a \in B$ then

$$a \cdot 1 = a$$

$$\& \ a \cdot 0 = 0$$

If we suppose that $0 = 1$ then the above 2 results are true only when $a = 0$. But we know that there are at least 2 elts in B & so

clearly $0 \neq 1$.

Now $0' = 0' + 0 = 1$ Axioms 4(a) & 6(a)

Axiom 4(b) & 6(b) - Exis of 1 & Compl.

(Identity)

(2) $1' = 1' \cdot 1$

$= 1 \cdot 1'$ - Comm

$= 0$ (Complementation)

Thm ~~[illegible]~~ **Involution Law**

In Boolean Algebra B, for each elt. $a \in B$ $(a')' = a$

Pf Let $a \in B$ then $a' \in B$

Also $a + a' = 1$ & $aa' = 0$ —— (1)

Now $a' \in B \Rightarrow (a')' \in B$

and $a' + (a')' = 1$ & $(a')(a')' = 0$ (Complementation)

Or $(a')' + a' = 1$ & $(a')' a' = 0$ — Comm —— (2)

from (1) & (2) $(a')' = a$ ie, the complement of a' is a.

Thm Cancellation Law

BA (J)

In the Boolean Algebra B, $\forall\, a, b, c \in B$

(1) $b+a=c+a$ & $b+a'=c+a' \Rightarrow b=c$

(2) $ba=ca$ & $ba'=ca' \Rightarrow b=c$

Pf. (1) Let $b+a=c+a$ & $b+a'=c+a'$

we have $b=b+0$

$=b+aa'$

$=(b+a)(b+a')$ — Dist

$=(c+a)(c+a')$ $(\because b+a=c+a)$

$\cancel{=(c+a)c+(c+a)a'}$

$=c+aa'$ — Distr

$=c+0$

$=c$

$\therefore b+c=c+a$ & $b+a'=c+a' \Rightarrow b=c$

(2) – By P. of duality.

De Morgan's Laws — ~~[illegible]~~ BA

In the Boolean Algebra B, $\forall\, a, b \in B$

(1) $(a+b)' = a'b'$

(2) $(ab)' = a'+b'$

Note: $+$ is join
$\cdot$ is meet

Pf. (1) We show $(a+b)+(a'b') = 1$ (Axiom 6 Existence Complement)

and $(a+b)(a'b') = 0$

Now, $(a+b)+(a'b') = [(a+b)+a'][(a+b)+b']$ (Distro.)

$= [a'+(a+b)][(a+b)+b']$ (Comm ~~Asso~~)

$= [(a'+a)+b][a+(b+b')]$ (Asso)

$= (1+b)(a+1)$

$= 1\cdot 1$ (Boundedness) $1+b=1$

$= 1$ —(1) (Idempotent)

Also,

BAB

$(a+b)(a'b') = a(a'b') + b(a'b')$ — Distr

$= (aa')b' + b(b'a')$ — Comm + Asso

$= (aa')b' + (bb')a'$ — Asso

$= 0 \cdot b' + 0 \cdot a'$ (Boundedness ~~Complement~~) Laws $a \cdot 0 = 0$

$= 0 + 0$ (idempotent)

$= 0$ —— (2)

~~To pro~~: From (1) & (2), $(a+b)' = a'b'$

(2) To prove: $a'+b'$ is the complement of ab, we show

$(ab) + (a'+b') = 1$ & $(ab)(a'+b') = 0$

Now, $(ab) + (a'+b') = (a'+b') + (ab)$

$= [(a'+b') + a][(a'+b') + b]$ (Distr)

$= [(a + (a'+b')][(a'+b') + b]$ (Comm + Asso)

$= [(a+a') + b'][a' + (b'+b)]$ — Asso.

BA F

$= (1+b')(a'+1)$ (Boundedness)

$= 1 \cdot 1$ (Boundedness) ~~(Idempotent)~~

idempotent

$= 1$ — (3)

Also, $(ab)(a'+b') = [(ab)a'] + [(ab)b']$

$= a'(ab) + (ab)b'$ — Comm

$= (aa')b + a(bb')$ — Asso

$= 0 \cdot b + a \cdot 0$

$= 0 + 0$ (Boundedness)

$= 0$ — (4) (idempotent)

From (3) & (4), $(a'b') = a'+b'$.

Order (or Inclusion) Relation on Boolean Algebra:-

Let $\langle B, +, \cdot, ', 0, 1 \rangle$ be a Boolean Algebra (BA). We define the binary relation "$\leq$" called the inclusion relation as follows:

$$a \leq b \iff ab' = 0 \text{ for } a, b \in B$$

Thm In the BA, $\langle B, +, \cdot, ', 0, 1 \rangle$, the binary relation "$\leq$" is a Partial Order Relation on B. We show:

(1) "$\leq$" is Reflexive

Since for each $a \in B$

$a' \in B$ & $aa' = 0$ — B is of Complementation

$\therefore aa' = 0 \Rightarrow a \leq a$ (def^n of $\leq$) $\forall a \in B$

$\therefore \leq$ is Reflexive

(2) $\leq$ is Antisymmetric

Let $a, b \in B$ & $a \leq b$, $b \leq a$

then $a \leq b \Rightarrow ab' = 0$

$b + ab' = (b+a)(b+b')$ — Distr

$\Rightarrow b + 0 = \cancel{(b+a)0}$ $(b+a) \cdot 1$ $(\because ab' = 0)$ $(b + b' = 1)$

$\Rightarrow \cancel{b+a+}$ $b = a + b$ — (1)

Fuc(2)

Again $b \leq a \Rightarrow b a' = 0$

$(a + ba') = (a+b)(a+a')$ - Distr

$\Rightarrow a + 0 = (a+b) \cdot 1$ $\quad (\because ba' = 0, a + a' = 1)$

$\Rightarrow a = a + b$ — (2)

from (1) & (2) $a = b$ (This equality is ordinary equality)

$\therefore a \leq b, b \leq a \Rightarrow a = b \Rightarrow$ "$\leq$" is Antisymmetric (defn of Antisymm.)

(3) $\leq$ is Transitive

Let $a \leq b$ & $b \leq c$ for $a, b, c \in B$

then $a \leq b, b \leq c \Rightarrow ab' = 0, bc' = 0$

$ac' = (ac') \cdot 1$ (identity)

$= (ac')(b + b')$

$= (ac')b + (ac')b'$ - Distr

$= a(c'b) + b'(ac')$ - Asso + Comm

$= a(bc') + (b'a)c'$ - Comm + Asso

$= a \cdot 0 + (ab')c'$ $\quad (\because bc' = 0)$

$= 0 + 0 \cdot c' = 0 \Rightarrow a \leq c$ (defn of $\leq$)

$\therefore a \leq b, b \leq c \Rightarrow a \leq c$

$\therefore \leq$ is Transitive

$\therefore \leq$ is Partial Order Rel on B.

Thm Let $\langle B, +, \cdot, ', 0, 1 \rangle$ be a Boolean Algebra & the partial order relation "$\leq$" be defined on B s.t. $a \leq b \iff ab' = 0$ for $a, b \in B$

Then

(1) $a+b = \sup\{a,b\}$ LUB

(2) $ab = \inf\{a,b\}$ GLB and

$(B, \leq)$ is a lattice. Also, the elts 1 & 0 of B are the supremum & infimum of this lattice. ($\Rightarrow$ 1 is universal upper bd - Greatest elt; 0 is univ. lower bd - Least elt of WHOLE lattice)

Pf. Let $a, b \in B$

(1) In order to show that $a+b$ is the sup of a & b, we first show that $a+b$ is an upper bound of a & b i.e., $a \leq a+b$ & $b \leq a+b$.

$\forall a, b \in B$

$$a(a+b)' = a(a'b') \quad \text{— De Morgan}$$
$$= (aa')b' \quad \text{— Asso}$$
$$= 0 \cdot b'$$
$$= 0$$

$\therefore a \leq a+b$ (defn of "$\leq$")

Similarly $b \le a+b$

$\therefore a+b$ is an upper bd of a & b. Now we shall show that $a+b$ is the sup. LUB of a & b.

Let c be any other upper bd. of a & b then $a \le c$ & $b \le c$

ie., $ac' = 0$ & $bc' = 0$

Now $(a+b)c' = c'(a+b)$ - Comm

$= c'a + c'b$ - Distr

$= ac' + bc'$ - Commutative

$= 0 + 0$

$= 0$

$\therefore a+b \le c$ (def^n of $\le$)

ie., any upper bound c of a & b is also an upper bd of $a+b$

$\therefore a+b$ is the supremum of a & b.

(2) First we shew that ab is the lower bd of a & b ie., $ab \le a$ & $ab \le b$

BA ③

Since $(ab)a' = (ba)a'$
$= b(aa')$
$= b \cdot 0$
$= 0$

$\therefore ab \leq a$ (defⁿ of $\leq$)

Similarly $ab \leq b$

$\therefore ab$ is a lower bound of a & b.

Now we ~~sho~~ show that ab is the infimum of a & b. Let c be any other lower bd of a & b, then

$c \leq a$ & $c \leq b$

i.e., $ca' = 0$ & $cb' = 0$

Now $c(ab)' = c(a'+b')$ — Demorgan
$= ca' + cb'$ — Dist
$= 0 + 0$
$= 0$

$\therefore c \leq ab$

i.e., any lower bound of a & b is also a lower bd of ab. $\therefore ab$ is the infimum of a & b

PTO Contd.

Consequently, $(B, \leq)$ is a lattice B.A(4)

where the operations $+$ & $\cdot$ are the joining meet $(\wedge)$ operations on B resp.

We now show that

$$a \leq 1 \text{ \& } 0 \leq a \quad \forall a \in B$$

Since $a \cdot 1' = a \cdot 0 = 0$ & $0 \cdot a' = 0$

~~defn of complement~~

$\therefore a \leq 1$ & $0 \leq a$ $\forall a \in B$ (defn of $\leq$)

$\therefore$ 1 & 0 are the supremum & infimum of B

HP.