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Thm 8.14 Let $\langle B, +, \cdot, ', 0, 1\rangle$ be a Boolean Algebra. Then $(B, \leq)$, where $a \leq b$ if & only if $ab' = 0$ for $a, b \in B$ is a distributive complemented lattice.

Pf. Let $\langle B, +, \cdot, ', 0, 1\rangle$ be a BA (Boolean Algebra). Then we know that $(B, \leq)$ where $a \leq b \iff ab' = 0$ for $a, b \in B$, is a bounded lattice & 1 & 0 are the (from previous Thm)

supremum & infimum of B resp.
Also, the binary operations + & · are the join ($\vee$) & meet ($\wedge$) op's on $(B, \leq)$ resp.

Since B is a BA, $\therefore \forall a, b, c \in B$

(i) $a(b+c) = ab + ac$
(ii) $a + (bc) = (a+b)(a+c)$ Distr Laws

Also for each $a \in B$

(i) $a + a' = 1$
(ii) $aa' = 0$ (Complementary Laws)

In above axioms, replacing + & · by $\vee$ & $\wedge$ resp., we get

(i) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$
(ii) $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

ie., $(B, \leq)$ is distributive &

(i) $a \vee a' = 1$ (ii) $a \wedge a' = 0$

ie., $(B, \leq)$ is complemented.

$\therefore (B, \leq)$ is a distributive complemented lattice.

~~Thm~~ Every distributive complemented lattice $(L, \leq)$ with $0 \neq 1$, determines a BA $\langle L, \wedge, \vee, ', 0, 1 \rangle$.

Pf Let $(L, \leq)$ be a distributive complemented lattice with $0 \neq 1$.

We know that in the lattice $(L, \leq)$

[L1] $a \vee b = b \vee a$ & $a \wedge b = b \wedge a$ $\forall a, b \in L$

Since $(L, \leq)$ is distributive complemented

$\therefore$ [L2] $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

& $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ $\forall a, b, c \in L$

[L3] For each $a \in L$, $\exists$ (there exists) a unique elt. $a' \in L$

s.t. $a \vee a' = 1$ & $a \wedge a' = 0$

where 1 & 0 are the supremum & infimum of L

Also, from defn of sup & inf,

[L4] $a \vee 0 = a$ & $a \wedge 1 = a$ $\forall a \in L$

Finally, it is given that

[L5] $0 \neq 1$

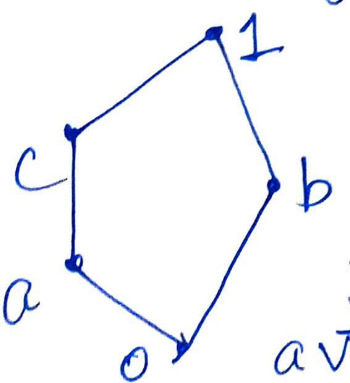
The properties from [L1] to [L5] shows that $\langle L, \wedge, \vee, ', 0, 1 \rangle$ is a Boolean Algebra

HP.

Cover of an element – Let a & b be 2 elements in a lattice. 'a' is said to ~~be~~ cover 'b' if $b < a$ and there is no element c s.t. $b < c < a$.

Atoms – Let L : lattice, with Least Element 0. An element $a \in L$ is called an **atom** if **a covers 0** i.e., $0 < a$ & there is no element c s.t. $0 < c < a$. **Atoms are immediately above 0.**

Irreducible Elements – Let L : lattice. An element $a \in L$ is said to be **join-irreducible** if $a = x \vee y \Rightarrow a = x$ or $a = y$. An element $a \in L$ is called **meet-irreducible** if $a = x \wedge y \Rightarrow a = x$ or $a = y$.



(c is not right above 0).

Atoms: a, b; c is NOT atom $\because 0 < a < c$ ($\because a < c$)

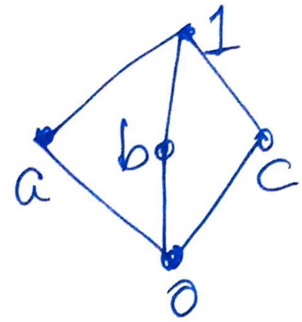
Also, c is Cover of a; 1 is Cover of both b & c [$\because c < 1$, $b < 1$ & no in-between elements].

Join-Irreducible elements: a, b, c

$a \vee 0 = a \Rightarrow a = a$ or $a = 0$; $c = c \vee a \Rightarrow c = c$ or $c = a$

$b \vee 0 = b \Rightarrow b = b$ or ...

Ex.



Atoms: a, b, c

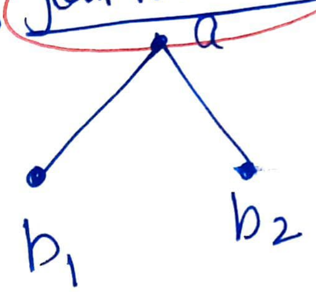
Join-irreducible: a, b, c

0 is join-irreducible

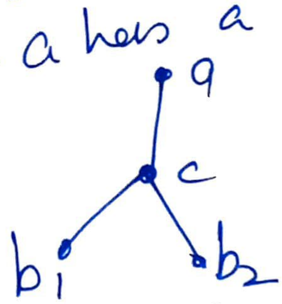
1 is NOT join irreducible

Atoms which immediately succeed 0 are join-irreducible. However, lattices can have other join-irreducible elements also

$a \neq 0$ is join-irreducible $\Leftrightarrow$ a has a unique immediate predecessor



(A)



(B)

(A): $a = b_1 \vee b_2$ so $a \neq b_1$ or $a \neq b_2$

Moreover, a has 2 immediate predecessor (elements below a) so a is NOT join-irreducible

(B): a has a unique immediate predecessor c

$\therefore a \neq LUB(b_1, b_2) = b_1 \vee b_2$

$\therefore$ a is join-irreducible.

Thm join (1)

<u>Thm</u> Let L: Complemented Lattice with unique complements. Then the join-irreducible elements of L other than 0 are its atoms.

<u>Pf.</u> Let L: Complemented Lattice with unique complements. Without loss of generality, assume L has more than 2 elements. If L contains 2 elements only then these 2 elements are 0 & 1 & 1 is clearly an atom. ($\because$ 1 is right above 0).

Let $a \neq 0$ be any join-irreducible element of L. <u>We prove: a is atom.</u>

Assume: a is NOT atom. $\Rightarrow \exists$ an element $x \in L$ s.t. $0 < x < a$ (x above 0) (def^n)

Let a': complement of a.

Now, $0 < x < a \Rightarrow x \wedge a' \leq a \wedge a'$

[$\because$ $0 \wedge a' < x \wedge a' < a \wedge a'$ by operating with $\wedge a$ throughout.] The $\leq$ sign is due to possibility of $=$ with a]

Thus join (2)

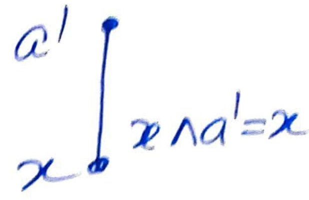
$\Rightarrow 0 \le x \wedge a' \le 0$ $\quad (\because a \wedge a' = 0)$

$\quad (\geq 0 \wedge a' = 0)$

$\Rightarrow x \wedge a' = 0$ —— (1)

(By Antisymmetry)

Now, if $x \le a'$ then $x \wedge a' = x$



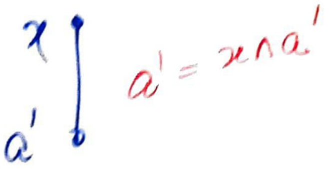
$\Rightarrow 0 = x$ —— from (1)

But this is NOT True ($\because 0 < x < a$)

$\therefore x \le a'$ is not possible

So, we suppose that $x \nleq a'$

If $a' \le x$



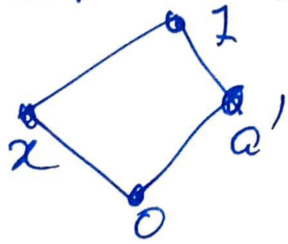
then by (1), $a' = x \wedge a' = 0$

$\Rightarrow a = 1$ $\quad (\because a' = 0 \Rightarrow a = 1)$

which is NOT join-irreducible in a Complemented lattice with more than 2 elements [$\because$ join-irred $\Rightarrow$ unique immediate predecessor]

If a' & x are NOT comparable

[ie, no line between them as in an example



]

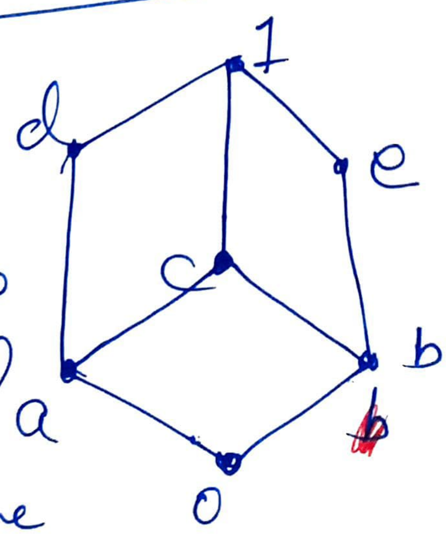
then $x \vee a' = 1$

$\because x \wedge a' = 0$ & $x \vee a' = 1$

$\Rightarrow x = a$ — Contradiction

$\therefore$ a is an Atom. H.P.

EX Lattice



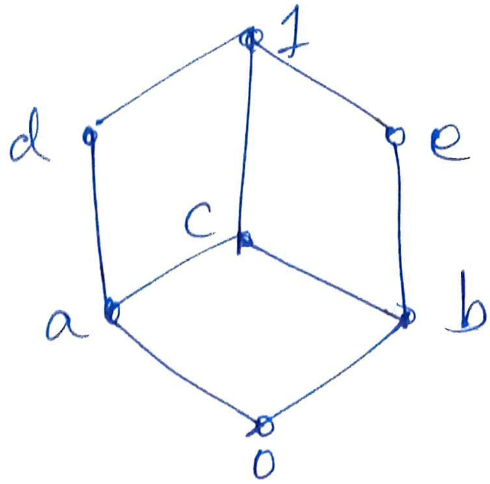
(a) which non-zero elts are join-irred?

(b) which elts are atoms?

Ans (a) join-irred: a, b, d, e ($\because$ d, e have unique immediate predecessor)

$c = a \vee b$ $\therefore$ c is NOT join-irr (No unique immediate predecessor)

(b) a, b: atoms ($\because$ immediately succeed 0)



(c) Find complements, if they exist for a, b, c.

Sol. Since $a \wedge e = 0$ and $a \vee e = 1$

$\therefore$ a & e are complements of each other

$b \wedge d = 0$ and $b \vee d = 1$

$\therefore$ b & d are complements of each other but c has no complement

(d) Is it a Complemented Lattice?

Sol. No $\because$ c has no complement.

(e) Is it a Complete Lattice?

Sol. Since L is finite, it is a Complete Lattice.

Ex Are there any Boolean Algebra with 3 elts? Why or why not?

Sol. No. Each Boolean Algebra must have 2^n elts.

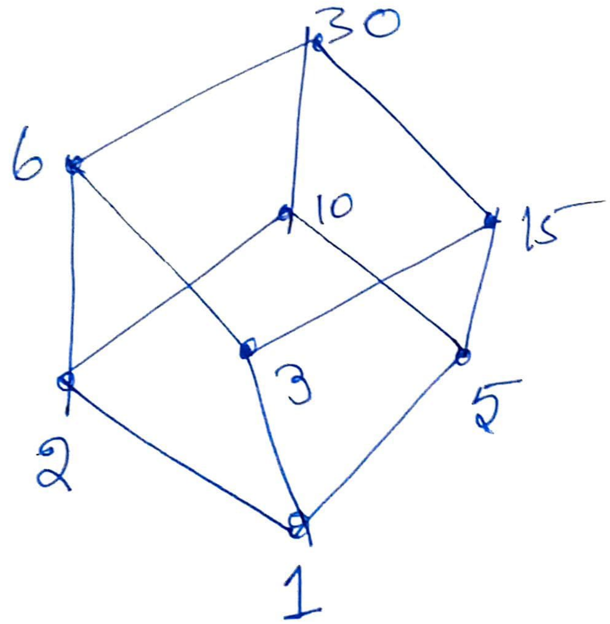
Ex: Consider the lattice $L = \{1, 2, 3, 5, 6, 10, 15, 30\}$ ordered by divisibility.

(a) which elts are join-irreducible?

(b) which elts are atoms?

(c) which elt is the greatest elt.?

Sol.

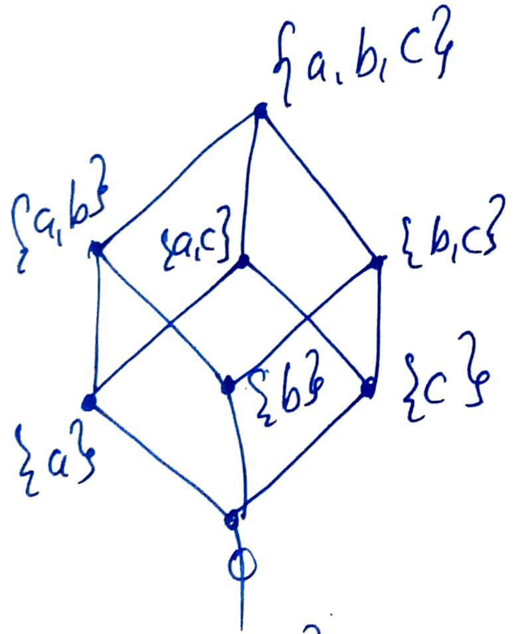


(a) Join-irreducible: elts with unique immediate predecessor $= 2, 3, 5$ (predecessor is 1 for 2, 3, 5)

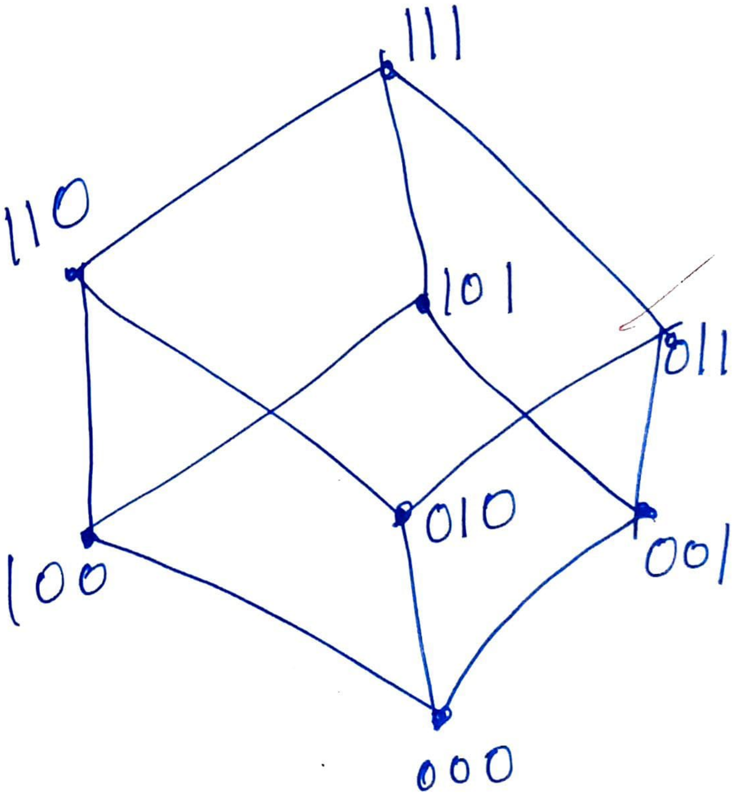
(b) Atoms: immediately above $0 = 2, 3, 5$ $(\equiv 1)$

(c) Greatest elt or 1 elt $= 30$.

Lattice B_n - If the Hasse Diagram of the lattice corresponding to a set with n elements is labelled by sequences of 0's & 1's of length n s.t. adjacent vertices differ in exactly one bit (0 or 1) then the resulting lattice is called B_n.

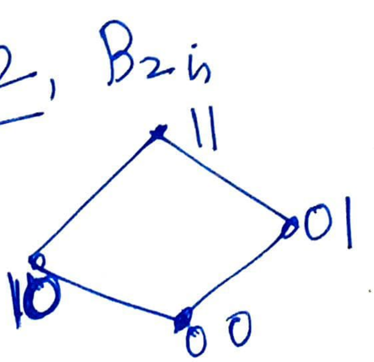


$(P(S), \subseteq)$
$S = \{a, b, c\}$

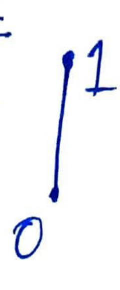


$n = 3 \Rightarrow B_3$

$n = 2$, B_2 is



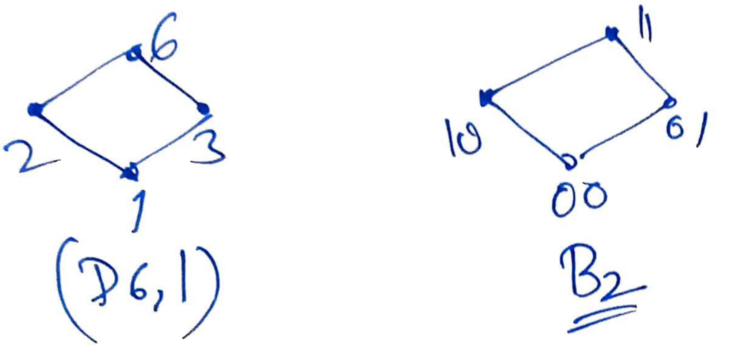
$n = 1$
B_1



$n = 0$

Each lattice $(P(S), \subseteq)$ is isomorphic to B_n where $n = |S|$ = No. of elements in set S. $(\cong)$ B_n ②

ex D_6 with partial order of divisibility



, $D_6 \cong B_2$
where $f(1) = 00$, $f(2) = 10$
$f(3) = 01$, $f(6) = 11$.

A finite Lattice is called a Boolean Algebra if it is isomorphic to B_n for some non-negative integer n. $\therefore$ each B_n is a Boolean Algebra & so is each lattice $(P(S), \subseteq)$ where S is a finite ~~set~~

$\therefore D6$ — also a Boolean Algebra.

Thm. Let $n = p_1 p_2 \cdots p_k$ (product of distinct primes) then D_n is a Boolean Algebra.

ex. $D_{30} = \{1, 2, 3, 5, 6, 10, 15, 30\}$ — 8 elements $= 2^n = 2^3$
$30 = 2 \cdot 5 \cdot 3$ (distinct primes $\therefore$ it is a BA)
D_{20}, $20 = 2 \cdot 2 \cdot 5$ (Not distinct primes $\therefore$ Not a BA)

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atoms, join irr (5)

Atoms & join-irreducible elts in Boolean Algebra -

\# Note: Only elts which are join-irred are those which cover the least elt. 0. Such elts are called Atoms of Boolean Algebra.

In a Boolean Algebra, except for the least elt. 0 & the atoms every other elt can be represented as the join of 2 or more atoms & this representation is unique.

Ex. Consider Boolean Algebra $(B_n, *, \oplus, ', 0, 1)$ ~~meet~~ join

Set $S = \{a_1, a_2, \ldots a_n\}$ of atoms of B_n.

Note that every elt of B_n except the atoms themselves & the least elt. 0 can be expressed as join of some of its atoms.

Show that there is an ~~1-1 correspondence~~ isomorphism between $(B_n, *, \oplus, ', 0, 1)$ and $(P(S), \cap, \cup, \sim, \phi, S)$.

$*$ meet, $\oplus$ join

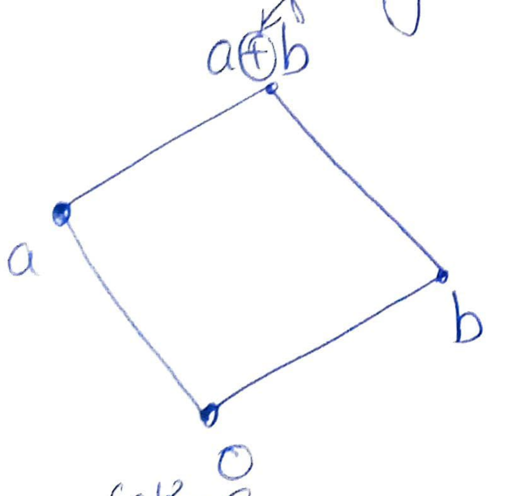
Sol Boolean Algebra $(B_n, *, \oplus, ', 0, 1)$

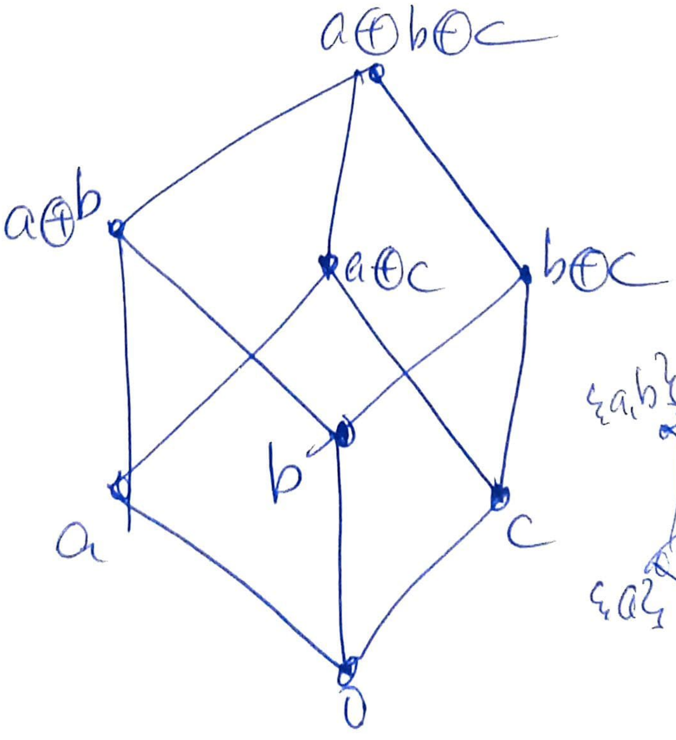
Set of atoms $S = \{a_1, a_2, \ldots, a_n\}$

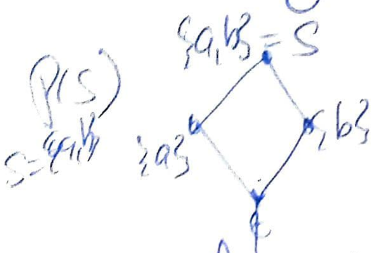
Consider: all possible subset of S

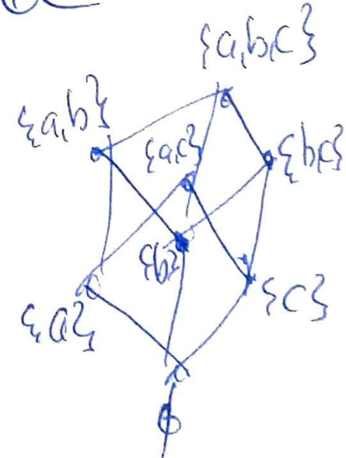
Associate with each subset of S — a Boolean expression of join of elts of the subset

$\Rightarrow$ an elt of B_n is associated with every subset with 2 or more elts of S









Associate elt $a_i \in B_n$ with subset $\{a_i\} \subseteq S$

$0 \in B_n$ with $\phi \subseteq S$. for ex., $a \oplus b$ with $\{a, b\}$

$\Rightarrow$ each subset of S is associated with a Boolean expression with join of elts of the subset and each such expression is ~~unique~~ distinct & represents an elt of B_n.

$\Rightarrow \exists$ 1-1 correspondence between

subsets of S & elts of B_n

& this 1-1 correspondence preserves the operations $*$, $\oplus$ and $'$.

$$\Rightarrow (B_n, *, \oplus, ', 0, 1) \underset{\text{isomorphic}}{\cong} (P(S), \cap, \cup, ', \phi, S)$$

This explains

$\Rightarrow$ why the diagrams of Boolean Algebras ~~are~~ given are n-cubes containing 2^n elts for $n = 1, 2, \ldots$

This also shows that a Boolean Algebra is completely specified by its Atoms.