Generally, $\forall x$ is very often followed by an implication. ($p \to q$)

$\exists x$ is very often followed by an AND (Conjunction)

ex. All mathematicians know Calculus.

Let $M(x)$: x is a mathematician

$C(x)$: x knows Calculus

$$\forall x, [M(x) \to C(x)] \quad (1)$$

(1) is the symbolic notation of the given ex.
(or mathematical representation).

ex Some parallelograms are squares.

Let predicates be

$P(x)$: x is a parallelogram

$S(x)$: x is a square.

Symbolic notation $\exists x, [P(x) \land S(x)]$

ex. Test the predicate for its Truth Value where it is given that

$$\forall x\,[q(x) \rightarrow s(x)]$$

where $q(x): x^2 \geq 0$

$s(x): x^2 - 3 > 0$

Sol. $x^2 - 3 > 0 \Rightarrow x^2 > 3$

$\&\ x^2 \geq 0$

Test for $x = 1$, $q(1)$ is T, $s(1)$ is F

$\because$ Using the Truth Table of Implication,

$q(1) \rightarrow s(1)$ i.e., $T \rightarrow F$

From the 2nd row, we get F

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

$\therefore \forall x\,[q(x) \rightarrow s(x)]$ is F.

$\because \exists x = 1$ s.t. $q(x) \rightarrow s(x)$ is F

$\therefore x = 1$ is the Counterexample which gives us the answer to be False.

Q
—

ex Determine whether the given predicates are T/F:

$s(x)$: " $2x$ is an even integer".

(i) $\forall x, s(x)$ — T

(ii) $\exists x \sim s(x)$ — F

$\sim$ NOT

ex. Test the predicate for its Truth Value where it is given that

$$\forall x\,[q(x) \rightarrow s(x)]$$

where $q(x) : x^2 \geq 0$

$s(x) : x^2 - 3 > 0$

Sol. $x^2 - 3 > 0 \Rightarrow x^2 > 3$

& $x^2 \geq 0$

Test for $x = 1$, $q(1)$ is T, $s(1)$ is F

$\because$ Using the Truth Table of Implication,

$q(1) \rightarrow s(1)$ i.e., $T \rightarrow F$

p	q	p→q
T	T	T
T	F	F
F	T	T
F	F	T

From the 2nd row, we get F

$\therefore \forall x\,[q(x) \rightarrow s(x)]$ is F.

$\because \exists x = 1$ s.t. $q(x) \rightarrow s(x)$ is F

$\therefore x = 1$ is the Counterexample which gives us the answer to be False.

ex Determine whether the given predicates are T/F:

$s(x)$: " $2x$ is an even integer".

(i) $\forall x\, s(x)$ — T

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$\sim$ NOT

Predicate Validity

Rules of Inference for Quantified Propositions

Rule 1: Rule of Detachment

$$\frac{\begin{array}{c} p \to q \\ p \end{array}}{\therefore q}$$

Rule 2: Transitive Rule

Rule 3: De Morgan's Laws

Rule 4: Law of Contrapositive

Rule 5: Universal Specification (US)

$$\frac{\forall x, P(x)}{\therefore P(c) \text{ for all } c}$$

Rule 6: Universal Generalization (UG)

$$\frac{P(c) \text{ for all } c}{\therefore \forall x, P(x)}$$

This Rule 6 is True provided $P(c)$ is True for ~~all~~ each element c in the Universe.

Rule 7: Existential Specification (ES)

$$\frac{\exists x, P(x)}{\therefore P(c) \text{ for some } c}$$

Note: element c is NOT arbitrary (as it was in Rule 5) but must be one for which $P(x)$ is True.

Rule 8: Existential Generalization (EG)

$$\frac{P(c) \text{ for some } c}{\therefore \exists x, P(x)}$$

Ex. Symbolize the following argument then check for its validity.

Lions are dangerous animals.
There are lions.

$\therefore$ There are dangerous animals.

Sol. Let $L(x)$: x is a lion
$D(x)$: x is dangerous

Argument becomes:

$$\frac{\begin{array}{l}\forall x, [L(x) \to D(x)] \\ \exists x, L(x)\end{array}}{\therefore \exists x, D(x)}$$

Note: Here Lions are dangerous animals, it is implied that All lions are dangerous animals.

Steps		Reasons
1.	$\exists x, L(x)$	Premise
2.	$L(a)$	Step 1, Rule 7 (ES)
3.	$\forall x, [L(x) \to D(x)]$	Premise
4.	$L(a) \to D(a)$	Step 3, Rule 5 (US)
5.	$D(a)$	Steps 2, 4, Rule 1 (Rule of Detachment)
6.	$\exists x, D(x)$	Step 5, Rule 8 (EG).

$\therefore$ Argument is Valid.

In problems like these, generally remove quantifiers properly, argue with the resulting propositions & then properly prefix the correct quantifiers.

EX. Prove: $(\exists x)(P(x) \wedge Q(x)) \rightarrow (\exists x)P(x) \wedge (\exists x)Q(x)$

Converse does not hold.

Sol.

	Steps	Reasons
1.	$(\exists x)(P(x) \wedge Q(x))$	Premise
2.	$P(y) \wedge Q(y)$	ES, step 1, y fixed
3.	$P(y)$	step 2, Rule of Conjunctive Simplification $[P \wedge Q \rightarrow P]$
4.	$Q(y)$	step 2, Rule of conjunctive Simplification $[P \wedge Q \rightarrow Q]$
5.	$(\exists x)P(x)$	step 3, EG
6	$(\exists x)Q(x)$	EG, step 4
7.	$(\exists x)P(x) \wedge (\exists x)Q(x)$	step 5, 6, Rule of Conjunction $P, Q \rightarrow P \wedge Q$

HP

Converse does not hold:

Steps	Reasons
1. $(\exists x) P(x) \wedge (\exists x) Q(x)$	Premise
2. $(\exists x) P(x)$	Step 1, Rule of Conjunctive Simplification $P \wedge Q \rightarrow P$
3. $\exists x Q(x)$	Step 2, Rule of Conjunctive Simplification $P \wedge Q \rightarrow Q$
4. $P(y)$	ES, Step 2, y is fixed
5. $Q(z)$	ES, Step 3

<u>Note</u> In step 4, y is fixed & it is no longer possible to use that variable again in step 5. We can not proceed any further to derive the LHS so converse does not hold.

HP.