Definitions -

Equivalence Class - Suppose R is an eq. val. rel. on set A. Let for each 'a' in A, [a] denote the set of elements of A to which 'a' is related under R i.e.,

$$[a] = [\, a \mid aRx \,]$$

Then we call [a] — the equivalence class of 'a' in A. & any element $b \in [a]$ is called a representative of the eq. val. class.

Quotient Set - The collection of all equivalence classes of elements of A under an equivalence rel. is denoted by A/R i.e.,

$$A/R = \{ [a] \mid a \in A \}$$

Quotient Set of A by the rel. R.

Ex– Also, $(P(S), \cup)$ is a Commutative Semigp.

Homomorphism of Semigps –

Let $(S, *)$ & $(W, \circ)$ be any 2 semigps.
Then a mapping $f: S \to W$ is called
a semigp homomorphism if $f(a*b) = f(a) \circ f(b)$ $\forall\, a, b \in S$

If f is bijective (1-1, onto) then it is called Isomorphism. If f is 1-1 then Monomorphism.
If f is onto then Epimorphism.

Similar definitions may be given for any algebraic structure with one binary operation.

Automorphism – A mapping from a Semigp $(S, *)$ onto itself which is homomorphism & bijective.

Note ① Homomorphism exists in all of the above.

② $f: S \to W$ is a Homomorphism, if f is also onto ~~there~~ & Domain$(f) = A$ [Called an everywhere defined fn if Domain$(f) = A$ in $f: A \to B$] then we say that W is a Homomorphic Image of S.

Typical Ex.

Ex Check for isomorphism of the foll.

Let $A = \{0, 1\}$. Consider the semigrps $(A^+, \cdot)$ and $(A, +)$ where $\cdot$ is the concatenation operation & $+$ is defined by the table

$+$	0	1
0	0	1
1	1	0

Define the functions $f : A^+ \to A$ by "

$$f(\alpha) = \begin{cases} 1 & \text{if } \alpha \text{ has an odd no. of 1's} \\ 0 & \text{if } \alpha \text{ has an even no. of 1's} \end{cases}$$

Sol. Homomorphism – Clearly if $\alpha, \beta \in A^+$ then $f(\alpha \cdot \beta) = f(\alpha) + f(\beta)$ is a homo (∵ of simple addition of no. of 1's).

Semgg 14

Onto: f is onto since

$f(0) = 0$ (even no of 1's) ~~is 0, or even~~ no.

$f(1) = 1$ (odd no of 1's)

f is NOT 1-1 ∵

Counterexample $f(110) = 0$ (even no of 1's)

$f(101) = 0$ (—do—)

Both have same no of 1's.

But $110 \neq 101$ ∴ NOT 1-1

∴ f is NOT an Isomorphism

Semigp (7)

Homework Exercise.

Ex Let alphabet $A = \{0, 1\}$ & consider the free semigp $(A^*, \cdot)$.
$\alpha R \beta \Leftrightarrow \alpha$ & β have the same no. of 1's.

Show that R is a Congruence Rel. on $(A^*, \cdot)$.

Hint: First show: R is equivalence Rel.
Same no. of 1's $\Rightarrow$ 110 & 101 have two (2) 1's. (Same no.) of 1's

ex. Semigp $(\mathbb{Z}, +)$ where $+$ is ordinary addition. Let $f(x) = x^2 - x + 2$. We now define the following rel. on $\mathbb{Z}$: $aRb \Leftrightarrow f(a) = f(b)$
Show: R is NOT a Congruence Rel.

Sol. R is an Eq. val. Rel on $\mathbb{Z}$ but NOT a Congruence Rel.

Counterexample: $f(-1) = (-1)^2 - (-1) - 2 = 0$
$f(2) = 0 \therefore -1 R 2$
Similarly $-2 R 3 \therefore f(-2) = f(3) = 4$

But $((-1) + (-2)) \not R (2 + 3)$
$\therefore f(-3) = 10, f(5) = 18 \therefore$ Not Congr. Rel.