Relation – A, B: non empty sets

A Relation from A to B is a subset of $A \times B$. (Cartesian product). If $R \subseteq A \times B$ & $(a,b) \in R$ we say a is related to b by R & write aRb. Else, $a \not R b$. A & B can be the same set also.

ex. $A = \{1,2,3\}$, $B = \{r,s\}$

$R = \{(1,r), (2,s), (3,r)\}$ is a relation from A to B.

ex. $A = \{1,2,3\}$, relation is "less than"

$aRb \iff a<b$

$R = \{(1,2), (2,3), (1,3)\}$

$1R2$ but $2 \not R 1$ $(\because 2 \not< 1)$

Types

Reflexive Relation – a relation on set A is Reflexive if $(a,a) \in R$ $\forall a \in A$ ie, if aRa $\forall a \in A$.

ex Let $R = \{(a,a) \mid a \in A\}$, R is relation of equality on set A.

$\because (a,a) \in R$ ie, $a = a$ $\forall a \in A$ $\therefore$ R is Reflexive.

Symmetric Relation – A rel. R on a set A is Symmetric if whenever aRb then bRa.

ex Let $A = \mathbb{Z}$ (set of integers) Rel 2

Let $R = \{(a,b) \in A \times A \mid a < b\}$

~~Counter~~ example $2 < 3 \not\Rightarrow 3 < 2$ $\therefore$ R is NOT symmetric.

~~Count~~ Counterexample

ex "=" is symmetric relation

Antisymmetric – If whenever aRb & bRa then $a = b$

OR

If whenever $a \neq b$ then $a \not R b$ or $b \not R a$.

ex. $A = \mathbb{Z}$

Let $R = \{(a,b) \in A \times A \mid a < b\}$

~~if $a < b \not\Rightarrow b < a$~~

If $a \neq b$ then either $a \not< b$ or $b \not< a$

$\therefore$ R is Antisymmetric.

ex, $A = \mathbb{Z}$, $R = \{(a,b) \in A \times A \mid a \text{ divides } b\}$

if $a|b$ and $b|a \Rightarrow a = b$ $\therefore$ R is Antisymmetric

ex $A = \{1,2,3,4\}$, $R = \{(1,2),(2,2),(3,4),(4,1)\}$

R is Antisymmetric $\because$ if $a \neq b$ then either $(a,b) \notin R$ or $(b,a) \notin R$, ex, $a=1, b=2$ & $(1,2) \in R$ but $(2,1) \notin R$ same for $(3,4), (4,1)$.

R is NOT Reflexive $\because$ $(a,a) \notin R$ $\forall a \in A$.

Transitive :- A rel. on set A is transitive if whenever aRb & bRc then aRc.

ex. Let $A = \mathbb{Z}$, R: "$<$"
If $a<b$ and $b<c \Rightarrow a<c$
$\therefore$ Transitive.

Equivalence Relation - Reflexive, Symmetric and Transitive.

EX Prove that congruence mod n is equivalence relation.

Sol. if $a \equiv b \bmod n$ then $a = qn + r$ & $b = tn + r$ (Division Algorithm)
& $(a-b)$ is a multiple of n.
$\therefore a \equiv b \bmod n \Leftrightarrow n \mid (a-b)$.

Reflexive $\quad a \equiv a \bmod n \Leftrightarrow n \mid (a-a) \Rightarrow n \mid 0$
$\therefore$ Reflexive $\forall a$

Symmetric $\quad$ if $a \equiv b \bmod n \Leftrightarrow n \mid (a-b)$
$nk = a-b$, $b-a = -nk = n(-k)$, $k \in \mathbb{Z}$
$\Rightarrow b \equiv a \bmod n \quad \therefore$ Symmetric

Transitive $\quad$ if $a \equiv b \bmod n$ & $b \equiv c \bmod n$
$\Rightarrow n \mid (a-b)$ & $n \mid (b-c)$
$\Rightarrow nk_1 = a-b$, $nk_2 = b-c$, $k_1, k_2 \in \mathbb{Z}$
Adding, $nk_1 + nk_2 = a-c$ or $nk_3 = a-c$ where $k_3 = k_1 + k_2 \in \mathbb{Z}$
$\Rightarrow n \mid (a-c) \Rightarrow a \equiv c \bmod n$
$\therefore$ Transitive $\therefore$ Equiv Rel.

Rel (4)

Ex. Show that R on set A is symmetric $\iff R = R^{-1}$. (R^{-1} is inverse relation)

Sol Let R be symmetric then for any $(a,b) \in R$, we have $(b,a) \in R$

$\Rightarrow (a,b) \in R^{-1}$ (def^n of R^{-1})

$\therefore R \subseteq R^{-1}$ ($\because (a,b) \in R$)

Let $(c,d) \in R^{-1}$ then

$(d,c) \in R$

$\Rightarrow (c,d) \in R$ ($\because R$ is symm.)

$\therefore R^{-1} \subseteq R$

$\therefore R = R^{-1}$

Converse Let $R = R^{-1}$

Let $(a,b) \in R$ then $(b,a) \in R^{-1}$

$\Rightarrow (b,a) \in R$ ($\because R = R^{-1}$)

$\Rightarrow R$ is symm.

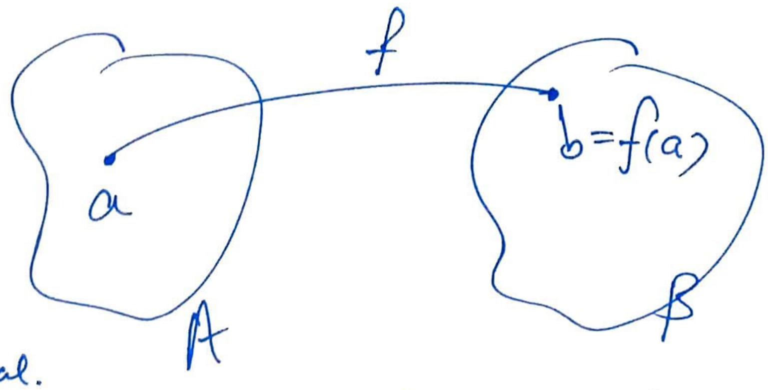
H.P.

Kolman p161 Fn (1)

Function – (mappings / transformations) – special type of Relation.

A f^n from a set A to set B is a relation from A to B, denoted by $f : A \to B$ s.t. $\forall a \in$ Domain(f) $f(a)$, the value or image of 'a' contains just one elt of B.

'a': argument



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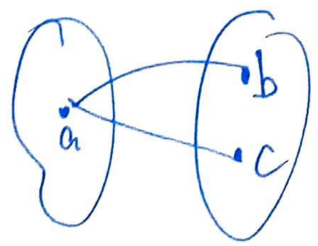
ex. The relation $f = \{(1,d), (2,c), (3,a)\}$ from $A = \{1,2,3\}$ to $B = \{a,c,d\}$ is a f^n from A to B, Domain(f) = A
Range(f) = B

ex. Relation $f = \{(a,b), (a,c), (b,d)\}$ from $A = \{a,b\}$ to $B = \{b,c,d\}$ is NOT a f^n!



$\because$ a is not single-valued

F[4] ②

ex $A=\{1,2,3\}$; $B=\{x,y,z\}$

Relation $R=\{(1,x),(2,x)\}$

$Dom(R) = \{1,2\}$

$Range(R) = \{x\}$

Special Types of f^ns: Let f be a f^n from A to B

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One-One (1-1) if $f(a)=f(a')$

$\Rightarrow a=a'$

OR if we cannot have $f(a)=f(a')$

for 2 distinct a & a' of A.

~~Surj~~

Onto:- if $Range(f)=B$.

Invertible f^n: $f:A\to B$ is invertible if its

inverse f^n, f^{-1} is also a f^n.

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ex $A=\{1,2,3,4\}$, $B=\{a,b,c,d\}$

$f=\{(1,a),(2,a),(3,d),(4,c)\}$

Then $f^{-1}=\{(a,1),(a,2),(d,3),(c,4)\}$

f^{-1} is NOT a f^n ∵ $f^{-1}(a)=\{1,2\}$ (not single valued)

∴ f is not invertible.

$f : Z \to Z$; Z (set of integers)

Here $Z = A$ & $Z = B$, ($f : A \to B$)

$f(n) = n+1 \ \forall n \in Z$

1-1? Yes $\because f(n) = f(m) \Rightarrow n+1 = m+1$

$\Rightarrow n = m$

Onto? Let b be an arbitrary element of $B (=Z)$.

~~Let b be an~~ Can we find an element $a \in A (=Z)$ s.t. $f(a) = b$?

Since $f(a) = a+1$, we need an element a in A s.t. (such that) $a+1 = b$

Since $a = b-1 \in A$, 'a' satisfies the eq^n

$\therefore$ Range$(f) = B$ $\therefore$ $\underline{f \text{ is onto}}$.

<u>Thm</u> Let $f : A \to B$ be a f^n. Then f^{-1} is a f^n from B to A $\Leftrightarrow$ f is 1-1.

<u>Sol.</u> Suppose f^{-1} is not a f^n. Then for some $b \in B$, $f^{-1}(b)$ must contain at least 2 distinct elements a_1 & a_2. Then $f(a_1) = b = f(a_2) \Rightarrow f$ is NOT 1-1.

<u>Converse</u> Suppose f is not 1-1. Then $f(a_1) = f(a_2) = b$ for 2 distinct elts a_1 & a_2 of A. $\therefore f^{-1}(b)$ contains both a_1 & a_2 so f^{-1} can not be a f^n. H.P.

Thm Let $f : A \to B$ be a f^n.

(a) Then f^{-1} is a f^n from B to A $\iff$ f is 1-1.

If f^{-1} is a f^n then

(b) the f^n, f^{-1} is also 1-1.

(c) f^{-1} is everywhere defined $\iff$ f is onto

(d) f^{-1} is onto $\iff$ f is everywhere defined.

Note: ~~Everywhere defined f^n~~ :-

If $f : A \to B$ is 1-1. Then f associates to each element 'a' of Domain(f) an element $b = f(a)$ of Range(f). Every b in Range(f) is matched in this ~~case~~ way, with 1 ~~&~~ only 1 element of Domain(f). For this reason, such an f is often called a <u>bijection</u> between Domain(f) & Range(f). If f is also <u>everywhere defined and onto</u>, then f is called a <u>one-to-one correspondence</u> between A & B

<u>Everywhere Defined f^n</u> :- $f : A \to B$.

Then f is <u>everywhere defined</u> if Domain(f) = A.

f is <u>onto</u> if Range(f) = B.

Pf. (a) We prove the following equivalent statement.

f^{-1} is NOT a f^n $\Leftrightarrow$ f is NOT 1-1.

Suppose first that f^{-1} is not a f^n. Then for some b in B, $f^{-1}(b)$ must contain at least 2 distinct elements: a_1 & a_2. Then $f(a_1) = b = f(a_2)$

so f is NOT 1-1.

Converse Suppose that f is NOT 1-1. Then $f(a_1) = f(a_2) = b$ for 2 distinct elements a_1 & a_2 of A. Thus $f^{-1}(b)$ contains both a_1 & a_2

so f^{-1} cannot be a f^n.

(b) Since $(f^{-1})^{-1}$ is ~~the~~ f^n, f, part (a) shows that f^{-1} is 1-1.

(c) We recall that Domain (f^{-1}) = Range(f)

Thus B = Domain (f^{-1}) $\Leftrightarrow$ B = Range(f)

In other words, f^{-1} is everywhere defined

$\Leftrightarrow$ f is onto.

(d) Since Range(f^{-1}) = Domain(f),

A = Dom(f) $\Leftrightarrow$ A = Range (f^{-1})

i.e. $\Rightarrow$ f is everywhere defined $\Leftrightarrow$ f^{-1} is onto.

Note: if $f: A \to B$ is 1-1 then $b = f(a)$ is equivalent to $a = f^{-1}(b)$.

LTP

Common

P166

ex. 16

R: Set of Real nos,

Let $f: R \to R$ be defined by

$f(x) = x^2$. Is f invertible?

Sol.

f is not onto

Range(f)

$\{x^2 : x \in R\}$

$= R^+ \cup \{0\} \neq R$, R^+: Set of +ve Real nos

Since $f(2) = f(-2) = 4$

$\Rightarrow f$ is not 1-1

$\therefore f$ is not invertible.

Bijective f^n — 1-1 & onto. Also called 1-1 correspondence. Domain$(f) = A$

Into: $f: A \to B$, B is also called Codomain.

↳ at least one element of codomain is not mapped by any element of domain. (i.e., NOT onto)

ex. $f: Z^+ \to Z^+$, $f(n) = n^2$ $\forall n \in Z^+$

$f: A \to B$

Into? Yes $\because$ 3 is not mapped by any element of $Z^+ (A)$

(Injective) 1-1? ~~No~~ Yes.

(Surjective) Onto? NO $\because 4 \in B$ but $(-2) \notin A = Z^+$

f^n (5)

ex $A = \{5, 6, 7\}$, $B = \{a, b\}$

$f : A \to B$ defined as $f(5) = a$, $f(6) = b$,

$f(7) = a$

Range$(f) = \{a, b\}$

1-1? NO $\because$ $f(5) = f(7) = a$
but $5 \neq 7$

Onto? Yes $\because$ Range$(f) = B$

Note: Every f^n is a Relation but every Relation is NOT necessarily a f^n!

ex. $A = \{1, 2, 3, 4\}$, $B = \{a, b, c\}$

$R = \{(1, a), (2, a), (3, a), (4, a)\}$

is both Relation & f^n from A to B.

But $S = \{(1, a), (2, b), (1, c), (3, a)\}$

is a Relation from A to B but NOT a f^n from A to B $\because$ the element 1 of A is associated with 2 different elements a & c of B.