Q (1)

Thm Let R be a Congruence relation on a Semigrp $(S, *)$. Then Quotient Set S/R is a semigrp wrt the operation $\oplus$ defined by

$$[a] \oplus [b] = [a * b] \quad \forall a, b \in S.$$

where $[a]$ denotes the equivalence class of element 'a' in S corresponding to the relation R.

Pf. Let $(S, *)$ be a semigrp & let R be a Congruence Relation on S. For any $a \in S$, the equivalence class $[a]$ is a set containing all those elements of S which are related to 'a' under the relation R. For any $a, b \in S$, $\oplus$ defined on S/R is

$$[a] \oplus [b] = [a * b]$$

$\oplus$ is well defined on S/R — Suppose $[a] = [a']$ & $[b] = [b']$. We need to show $[a] \oplus [b] = [a'] \oplus [b']$ i.e., $[a * b] = [a' * b']$

Since $[a] = [a']$ & $[b] = [b'] \Rightarrow aRa'$ & bRb'.

Since, R is a Congruence Relation, Q(2)

$\therefore aRa'$ & $bRb' \Rightarrow a*b\,R\,a'*b'$

$\Rightarrow [a*b] = [a'*b']$

$\Rightarrow \oplus$ is well defined on S/R

$\Rightarrow \oplus$ is a Binary Operation on S/R

$\oplus$ is Associative – For any $a, b, c \in S$,

$[a] \oplus ([b] \oplus [c]) = [a] \oplus [b*c]$

$= [a*(b*c)]$

$= [(a*b)*c]$ ($\because *$ is Asso on S)

$= [(a*b)] \oplus [c]$

$= ([a] \oplus [b]) \oplus [c]$

$\Rightarrow \oplus$ is Asso & $\therefore S/R$ is a Semigp

The Semigp $(S/R, \oplus)$ is called Quotient Semigp or Factor Semigp of Semigp $(S, *)$ (of the Congruence Rel. R.

$\oplus$ is sometimes called Quotient Binary Relation

Corollary. Let R be a Congruence Relation on monoid $(M, *)$. Then $(M/R, \circledast)$ is a Monoid, where $\circledast$ on M/R is defined by

$$[a] \circledast [b] = [a * b]$$

Pf Let $(M, *)$ be a Monoid with identity e. Then by above Thm., $(M/R, \circledast)$ is a Semigrp. We show $(M/R, \circledast)$ has an identity element. Let $[a]$ be any element in M/R where $a \in M$. Then

$$[a] \circledast [e] = [a * e]$$
$$= [a] = [e * a]$$
$$= [e] \circledast [a]$$

$\Rightarrow [e]$ is the identity of M/R.

$\therefore (M/R, \circledast)$ is a Monoid.

Thm Let $(S, *)$ be a Semigrp & let R be a Congruence Relation on $(S, *)$. Then Prove that $\exists$ a homomorphism from $(S, *)$ onto Quotient Semigrp $(S/R, \circledast)$.

Pf. We define a f^n; $f: S \to S/R$ as follows:

$f(a) = [a]$, the equivalence class of $a \in S$ under R. We show that f is onto & homomorphism

f is onto: Let $[a]$ be any element in S/R. Then $a \in S$ & $f(a) = [a]$.

$\Rightarrow f$ is onto.

f is homomorphism — Let a, b be any 2 elements of S. Then

$$f(a+b) = [a+b]$$
$$= [a] \oplus [b] \quad (\text{def}^n \text{ of } \oplus)$$
$$= f(a) \oplus f(b) \quad (\text{def}^n \text{ of } f)$$

$\therefore f$ is a homomorphism.

The mapping $f: S \to S/R$ defined by $f(a) = [a]$ is called <u>Natural Homomorphism.</u>

Since, R is a Congruence Relation, Q②

$\therefore a R a'$ & $b R b' \Rightarrow a * b \, R \, a' * b'$

$\Rightarrow [a*b] = [a' * b']$

$\Rightarrow \oplus$ is well defined on S/R.

$\Rightarrow \oplus$ is a Binary Operation on S/R.

$\oplus$ is Associative — For any $a, b, c \in S$,

$[a] \oplus ([b] \oplus [c]) = [a] \oplus [b*c]$

$= [a * (b*c)]$

$= [(a*b) * c]$ $\quad (\because *$ is Asso. in $S)$

$= [(a*b)] \oplus [c]$

$= ([a] \oplus [b]) \oplus [c]$

$\Rightarrow \oplus$ is Asso & $\therefore S/R$ is a Semigrp.

The Semigrp $(S/R, \oplus)$ is called Quotient Semigrp or Factor Semigrp of Semigrp $(S, *)$ by the Congruence Rel. R.

$\oplus$ is sometimes called Quotient Binary Relation