Thm   Fundamental Thm of Homomorphism, of Semigroups.   Fund (1)

Let $(S, *)$ & (T, o) be semigrps. If $f: S \to T$ is a semigrp homomorphism, then semigrp (T, o) is isomorphic to some Quotient Semigrp of $(S, *)$.

Pf. Let $f: S \to T$ be a semi grp homomorphism of the semigrp $(S, *)$ onto the semigrp (T, o). Let R be the congruence Relation on $(S, *)$ corresponding to the homomorphism f. Then R is defined by

$aRb \iff f(a) = f(b) \ \forall a, b \in S.$

We show: (T, o) is isomorphic to Quotient Semigrp $(S/R, \circledast)$ of $(S, *)$.

If $a \in S$ then $[a] \in S/R$ & $f(a) \in T$.

Let us define a mapping

$\psi: S/R \to T$ by setting $\psi([a]) = f(a) \ \forall a \in S.$

We show: ψ is a semigp homomorphism. Fund (2)

ψ is well defined :-

Suppose that $a, a' \in S$ & $[a] = [a']$

$[a] = [a'] \Rightarrow aRa' \Rightarrow f(a) = f(a')$

$\Rightarrow \psi([a]) = \psi([a'])$

$\therefore \psi$ is well-defined

ψ is 1-1 : Suppose $\psi([a]) = \psi([b])$

$\Rightarrow f(a) = f(b) \Rightarrow aRb \Rightarrow [a] = [b]$

$\therefore \psi$ is 1-1.

ψ is onto : Let b be any element of T. Since f is onto, $\exists$ an element $a \in S$ s.t. $b = f(a)$. Now $\psi([a]) = f(a) = b$ $\therefore \psi$ is onto.

ψ is semigp homomorphism - Let $[a], [b]$ be any 2 elements in S/R. Then

$\psi([a] \circledast [b]) = \psi([a * b]) = f(a * b)$ (def^n)

$= f(a) \circ f(b)$ ($\because f$ is homo.)

$= \psi([a]) \circ \psi([b]) \Rightarrow \psi$ is an Isomorphism of S/R onto T. $\therefore (S/R, \circledast) \cong (T, \circ)$ HP

Thm. If f be an isomorphism of a semigrp S_1 onto a semigrp S_2 then f^{-1} is also an isomorphism of S_2 onto S_1.

Pf. Since f is bijective ($\because f$ is iso.) so f^{-1} is also bijective where $f^{-1}: S_2 \to S_1$ is defined as $f^{-1}(b) = a$ whenever $f(a) = b$ [f is Iso. $\Rightarrow f^{-1}$ is also Iso.].

for $a \in S_1$ & $b \in S_2$.

We show: f^{-1} is a homomorphism.

Let $a_1, b_1 \in S_1$ & $a_2, b_2 \in S_2$ be s.t.

$f(a_1) = a_2$ s.t. $f^{-1}(a_2) = a_1$

& $f(b_1) = b_2$ s.t. $f^{-1}(b_2) = b_1$

Now $f^{-1}(a_2 b_2) = f^{-1}(f(a_1) f(b_1))$

$= f^{-1} f(a_1 b_1) = a_1 b_1$ ($\because f$ is Iso.)

$= f^{-1}(a_2)\, f^{-1}(b_2)$

$\Rightarrow f^{-1}$ is a homo. & bijective $\Rightarrow f^{-1}$ is Iso.

H.P.

Homework Exercise

Ex Show that Semigp $(Z, +)$ is isomorphic to semigp $(E, +)$ where E is set of even integers.

Hint: define a mapping $\phi : Z \to E$ by $\phi(n) = 2n$, $n \in Z$.

Ex Let S be a semigp & F be the semigp of functions from S to itself. Show: there is a homomorphism $\phi : S \to F$ defined by
$\phi(a) = f_a \in F \ \forall a \in S$.

Pf For any $a \in S$, Let $f_a : S \to S$ be defined by
$f_a(b) = ab$ for $b \in S$

Now $f_{ab}(c) = (ab)c = a(bc)$ ($\because$ S is Semigp)

$= f_a(bc) = f_a(f_b(c))$ (def^n)

$= f_a f_b(c) \quad \forall c \in S$

$\therefore f_{ab} = f_a f_b$

$\therefore \phi(ab) = f_{ab} = f_a f_b = \phi(a)\phi(b) \quad \forall a, b \in S$

$\therefore \phi$ is a Homo. HP