B.Sc pt- III

Paper - II - Electrodynamics

Unit - I

Electromagnetic Induction

It shows that how a time varying field (magnetic) gives rise to an electric field and vice-versa.

Electromotive force (e.m.f.)

For steady current flow in conductor, a potential difference must exist b/w its two ends.

This is maintained by a source of e.m.f.

The devices provide the e.m.f. are

(i) battery,

(ii) Dynamo etc.

They convert some other kind of energy into electrical energy; example

Battery → chemical energy → electrical energy

Dynamo → mechanical energy → electrical energy

Let If Battery maintains the potential difference of V volt b/w terminals then its e.m.f. is V volt.

The line integral of the field over a path b/w two points (A, B) is equal to the potential difference the points (if it is terminals of the battery) the e.m.f of battery

$$V = \int_A^B \vec{E} \cdot \vec{dl} \quad \text{——— (1)}$$

When current flow in conductor, the battery must do work to keep the potential difference b/w its terminal constant. If a charge q moves from one terminal of battery to other through the conductor.

The work done by battery is

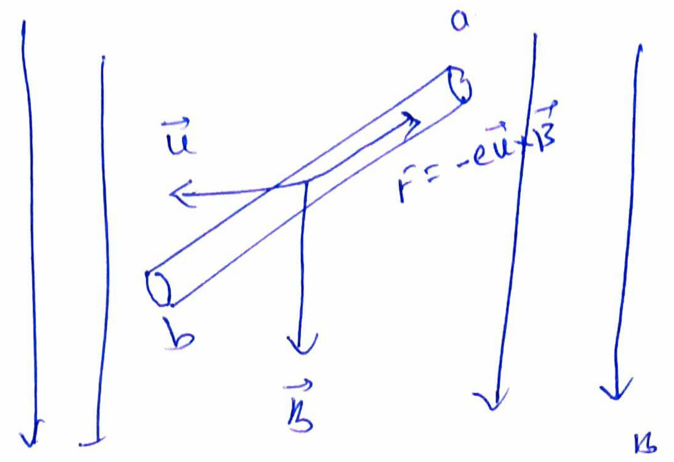
$$Vq = \int_A^B q\vec{E} \cdot \vec{dl}$$

The e.m.f. of battery

$$V = \frac{1}{q} \int_A^B q\,\vec{E} \cdot \vec{dl} \quad \text{——— (2)}$$

The e.m.f. is not force but it is a quantity which has the dimensions energy per charge.

Consider a metal bar of length 'L' moving with velocity 'v' in perpendicular direction of uniform magnetic field B.



Then electrons of bar move towards the end of bar and collect them setting a field given by

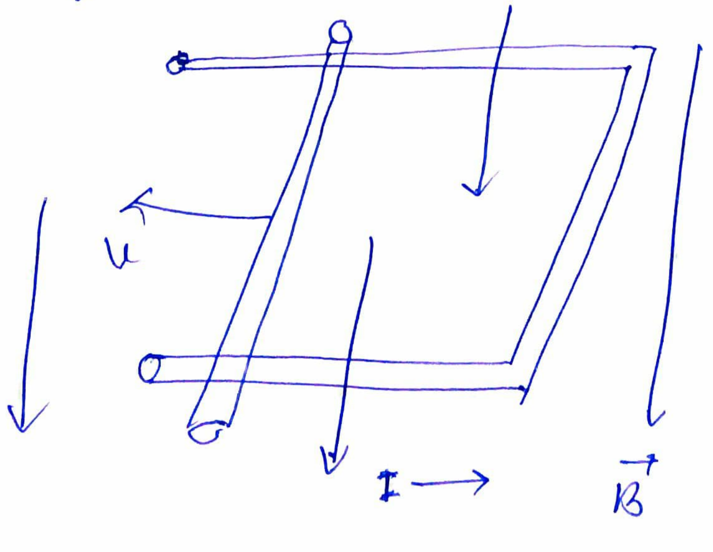
$$\vec{E} = -\vec{u} \times \vec{B} \quad —(3)$$

The potential difference b/w end of bar is

$$V_{ba} = \int_b^a \vec{E} \cdot \vec{dl} = uBL \quad —(4)$$

The potential difference produces no electric current flow.

Again if bar forms a part of a circuit then there will flow of current.



The line integral of force on charge q round the circuit

$$\oint q\vec{E}\cdot d\vec{l} = quBL$$

The e.m.f. induced in the closed circuit due to motion of conductor is

$$V = \frac{1}{q}\int q(\vec{E}\cdot d\vec{l}) = uBL \quad (5)$$

This is called motional e.m.f. since it depends on velocity of conductor.

We have two sorts of voltage

(i) electrostatic potential difference due to stationary charge

(ii) electromotive force due to moving charge.

Here uBL is the magnetic flux through the area swept by the bar in unit time

The rate of charge of flux through the circuit is uBL;

$$|e.m.f| = uBL = \frac{d\phi}{dt} \quad (6)$$

where ϕ is total magnetic flux.

Faraday's Law of Electromagnetic Induction

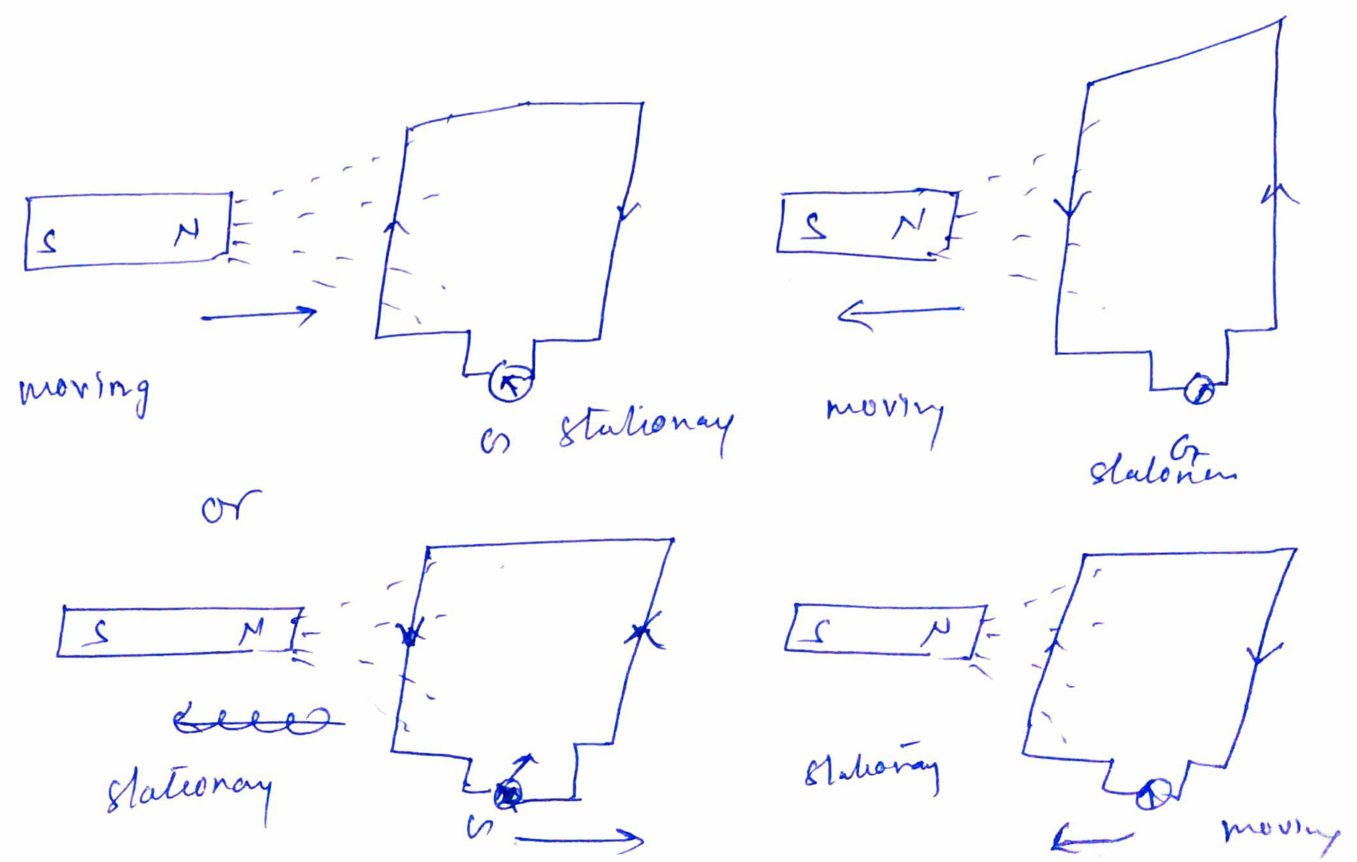
Faraday and Henry observed that

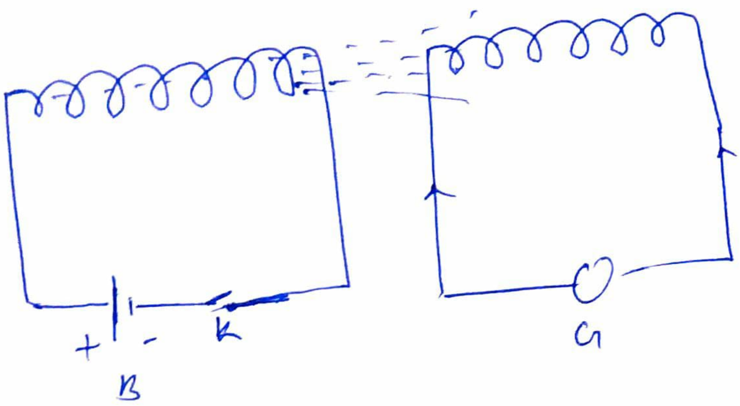
(i) If a magnet is moved about to the neighbourhood of a wire formed closed circuit (with no battery), then a current is produced in the wire so long as the movement lasts.

(ii) The same effect is observed if the magnet is kept still and the circuit is moved.

This gives ~~that~~ an impression that for production of current in the circuit, there must be a relative movement.

(iii) A transient current is induced in a loop of wire if the stationary current in an adjacent circuit is turned on or off





Either on or off the switch

or movement of coil.

The changing flux induces an electric field and hence an e.m.f. in the circuit which causes the current to flow

Faraday called this phenomenon electromagnetic induction

Faraday's results were summed up in what is known as the flux rule.

The changing magnetic flux induces e.m.f. in circuit ~~the magnetic field then~~ ~~since flux is proportional to~~
The magnitude of e.m.f is given by
If ε is e.m.f and ϕ is flux then

$$|\varepsilon| \propto \frac{d\phi}{dt} \quad (1)$$

The direction of induced e.m.f. is provided by Lenz's law

Lenz's law :-

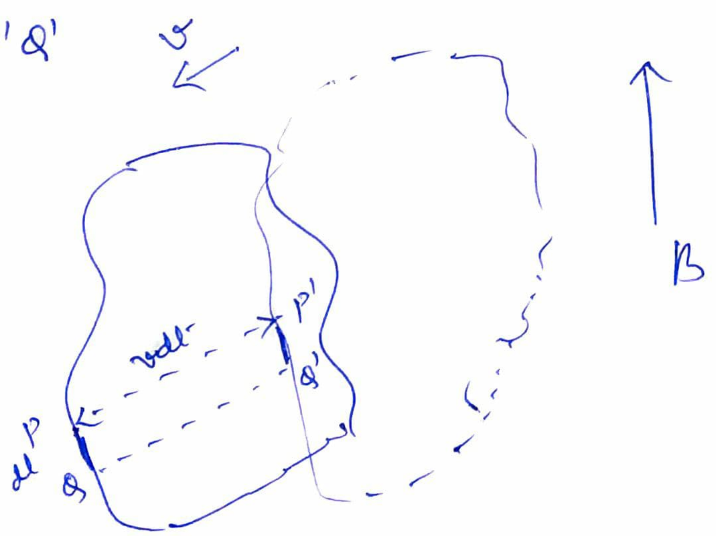
The direction of induced e.m.f is such that the magnetic flux associated with the current generated by it, ~~opposite~~ opposes the original change of flux causing the e.m.f.

$$\varepsilon = -\frac{d\phi}{dt} \propto -\frac{dB}{dt}$$

Induction law of moving circuit

Consider a circuit of an arbitary ~~space~~ shape moving in a time-independent magnetic field '$\vec{B}$' with velocity '$\vec{v}$' (not uniform)

~~the~~ An Element dI (PQ) will moves in time dt a distance $v\,dt$ to position P'Q'



Let 'u' is velocity of conducting electrons relative to wire the velocity relative to field is 'v+u'

The fore exerted on each electron is

$$e(\vec{v}+\vec{u}) \times \vec{B}$$

The component of this force along the element dl is

$$\{ e(\vec{v}+\vec{u}) \times \vec{B} \} \cdot \hat{e}_l$$

where $\hat{e}_l$ is unit vector in PQ direction

Since $\hat{e}_l$ is parallel to 'u' then $(u \times B) \cdot \hat{e}_l = 0$

Then

$$(e(\vec{v}+\vec{u}) \times \vec{B}) \cdot \hat{e}_l = (e\vec{v} \times \vec{B}) \cdot \hat{e}_l \quad — (1)$$

This shows that there is an electric field $\vec{E} = \vec{v} \times \vec{B}$ induced in the wire,

its component along the wire is $(v \times B) \cdot \hat{e}_l$

The induced e.m.f is

$$\mathcal{E} = \oint (v \times B) \cdot \hat{e}_l \, dl \quad — (2)$$

Since circuit moves, the element of circuit dl ~~swept~~ sweeps in dt an area ~~PQQ'~~ $PQQ'P' = v\,dt \times \hat{e}_l dl$

The flux across this element is $(v\,dt \times \hat{e}_l dl) \cdot B$

Then total flux $d\phi = \oint B \cdot (v\,dt \times \hat{e}_l dl)$

$$\frac{d\phi}{dt} = \oint B \cdot (v \times \hat{e}_l dl) = -\oint (v \times B) \cdot \hat{e}_l dl \quad — (3)$$

Comparing eqn (2) and (3) $\quad \mathcal{E} = -\frac{d\phi}{dt} \quad — (4)$

Integral and Differential form of Faraday's law

The induced e.m.f. is equal to the

$$\varepsilon = \oint \vec{E}\cdot\vec{dl} \quad \text{--- (1)}$$

where $\vec{E}$ is electric field
$\vec{dl} = \hat{e}_l dl$ displacement.

The magnetic flux through coil is

$$\phi = \int_s \vec{B}\cdot\hat{e}_n ds$$

$$\phi = \int_s \vec{B}\cdot\vec{ds} \quad \text{--- (2)}$$

where $\vec{B}$ is magnetic field
$\vec{ds}$ = surface Area of coil

Then we know from Faraday's law

$$\varepsilon = \oint \vec{E}\cdot\vec{dl} = -\frac{d}{dt}\phi = -\frac{d}{dt}\int \vec{B}\cdot\vec{ds} \quad \text{--- (3)}$$

Since surface area 'S' does not depend on time then

$$\oint \vec{E}\cdot\vec{dl} = -\int_s \frac{d\vec{B}}{dt}\cdot\vec{ds} \quad \text{--- (4)}$$

This is the integral form of Faraday's law

Using Stoke's theorem

$$\oint \vec{E}\cdot\vec{dl} = \int_s \text{Curl}\,\vec{E}\cdot\vec{ds} = \int (\nabla\times\vec{E})\cdot\vec{ds} \quad \text{--- (5)}$$

Using eqn (4) and (5), we get

then $\int_S \text{curl}\,\vec{E}\cdot d\vec{s} = -\int_S \frac{d\vec{B}}{dt}\cdot d\vec{s}$

then

$$\int \left(\text{curl}\,\vec{E} + \frac{\partial B}{\partial t}\right)\cdot ds = 0 \quad \text{—— (6)}$$

Since $d\vec{s}$ is arbitary so $d\vec{s} \neq 0$, then

$$\text{curl}\,\vec{E} + \frac{\partial B}{\partial t} = 0$$

$$\text{curl}\,\vec{E} = \nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \quad \text{—— (7)}$$

this is differential form of Faraday's law of e.m. induction

we also know that

$$\text{div}\,\vec{E} = \nabla\cdot\vec{E} = \rho/\epsilon_0 \quad \text{—— (8)}$$ (Gauss' law in differential form)

Eq^n (7) and (8) show that $\vec{E}$ has a non-conservative part due to changing $\vec{B}$ as well as conservative part due to ρ (charge density)

the sources of e.m. field are two kinds

(i) first kind is associated with a system ~~involved~~ such as Electrostatics in which energy is conserved during cyclic process.

(ii) this source of such type (conservative or irrotational) is described by $\nabla\cdot\vec{E} = \rho/\epsilon_0$

where $\vec{D} = \cancel{\phi}$

$\vec{E} = -\nabla V$ $V \; \phi \rightarrow$ Electric ~~flux~~ potential

$$\text{Curl}(\text{div} E) = 0$$

fun

$$\text{Curl grad } V = 0$$

$$\nabla \times (\nabla V) = 0 \quad \text{— (9)}$$

(ii) The second kind of source is associated with a system in which there is energy transferred in a cyclic process (i.e. magnetic field of a solenoid)

and specified by curl source and has no divergence

Therefore any vector field is uniquely determined if its div. and curl sources are given

This is known as Helmholtz theorem

Taking div. of eqn (7)

$$\nabla \cdot (\nabla \times \vec{E}) = -\frac{\partial}{\partial t}(\nabla \cdot \vec{B}) = 0$$

This is true only if $\nabla \cdot \vec{B} = 0$ — (11)   { $\nabla \cdot B$ is independent of time at every point)

i.e. $\vec{B}$ is always solenoidal

The Faraday's law has two important consequences:-

(i) $\vec{E}$ is no loger a conservative field when $\vec{B}$ varies with time. It indicates that energy can flow b/w electric and magnetic forms through time varying field.

(ii) No free magnetic pole can exist. All magnetic poles occur in pairs (positive and negative)

Faraday's law of induction show that $\vec{E}$ and $\vec{B}$ are not independent when we consider their time-dependence. Therefore these two field would be considred as a single field:- e.m. field.

<u>self-and Mutual inductances</u>

when current I flows in a circuit, there will be a magnetic flux(ϕ)

the flux is proportional to I

$$\phi \propto I$$

$$\phi = LI \quad \text{—} \; (1)$$

where L is constant

Consider if ϕ changes with time i.e. I is time dependent

then $\phi = \phi(t)$ function of time

then

Therefore

$$\frac{d\phi}{dt} = \frac{d\phi}{dI} \cdot \frac{dI}{dt} = L\frac{dI}{dt} \qquad \text{from eq}^n \ 1$$

then

$$L = \frac{d\phi}{dI} \quad — \ 2$$

The L is called self inductance

induced e.m.f.

$$\mathcal{E} = -\frac{d\phi}{dt} = -L\frac{dI}{dt} \quad — \ 3$$

L depends on the geometry of the circuit as well as permeability of the medium.

self induction is defined as: a total flux through the circuit when unit current is flowing

$$\phi = L \text{ when } I \text{ is unit}$$

$$L = \phi \quad (\text{total flux})$$

If $I = 1$ Ampere

when $L = \phi$ (weber) — 4

Again when rate of change of current is 1 Ampere/sec then

$$\mathcal{E} = -L$$

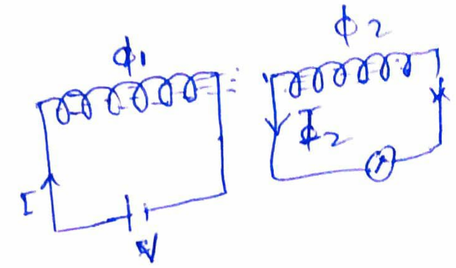
$$L = -\mathcal{E} \Rightarrow L = \mathcal{E} \ ~~(volts)~~ \ (\text{Henry})$$

— 5

The circuit has unit self inductance if $\mathcal{E} = 1$ volt

Let first coil carries a ~~constant~~ current I_1 and second coil is brough near this coil, then will be ϕ_2 through second coil due to the current in first coil

Since $\phi_2 \propto I_1$



$$\phi_2 = L_{21} I_1 \quad \text{—— (6)}$$

where L_{21} is constant

There will also be a flux through first circuit due to I_2 in second coil. then

$$\phi_1 = L_{12} I_2 \quad \text{—— (7)}$$

where L_{12} is again a constant

we know that potential energy of system is given as

$$U_p = -\phi_2 I_2 = -L_{21} I_1 I_2$$
$$= -\phi_1 I_1 = -L_{12} I_2 I_1$$

$\left\{ \begin{array}{l} U_e = -\vec{p}\cdot\vec{E} \\ U_B = -\vec{M}\cdot\vec{B}\cdot ds \\ U_B = I\phi \Rightarrow \Phi = M\cdot ds \end{array} \right.$

Therefore $L_{12} = L_{21}$ —— (8)

Then $L_{12} = L_{21} = M$ is called mutual inductance b/w two coils

we know that

$$U_p = -\phi_1 I_1 = -I_1 \int \vec{B}_1 \cdot d\vec{s}$$

$$U_p = -I_1 \int (\nabla \times A_{12}) \cdot ds$$

$\{B = \nabla \times A$

$$= -I_1 \int \vec{A}_{12} \cdot \vec{dl}_1$$

using stokes's theorem

$A \rightarrow$ magnetic vector potential

$$= -I_1 \int \left(\frac{\mu}{4\pi} \int \frac{I_2 dl_2}{|r|} \right) \cdot \vec{dl}_1$$

$$= -\frac{\mu}{4\pi} I_1 I_2 \int \int \frac{\vec{dl}_1 \cdot \vec{dl}_2}{|r|}$$

$$U_p = -L_{12} I_1 I_2$$

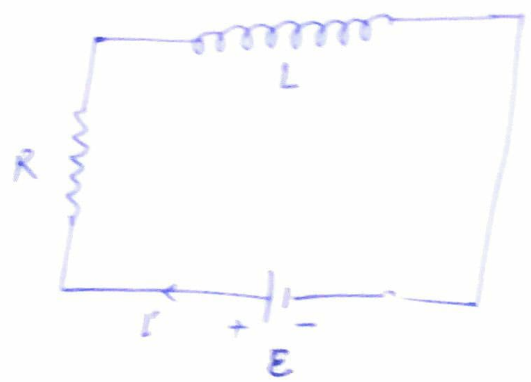
This is known as Neumann formula

Unit of M is Henry.

Energy ~~stored~~ in Magnetic fields

The energy in a field is the total work done to establish it. The steady current produces the magnetic field. We know that the time dependent field (current or flux) will induce the e.m.f. in the ~~circuit~~ circuit. Therefore it should be considered while calculating the energy.

(a) Magnetic energy stored in a inductor



Consider a simple circuit shown in figure.

Let $I(t)$ is the current in the circuit at time 't'

The voltage drops across the L is $L\frac{dI}{dt}$

The net forward e.m.f is $\mathcal{E} - L\,dI/dt$

By Ohm's law

$$\mathcal{E} - L\frac{dI}{dt} = RI \quad \text{——} (1)$$

Consider workdone by e.m.f. $\mathcal{E}$ in moving a small amount of charge dQ through the circuit

$$dW = \mathcal{E}\,dQ \qquad \{ U = qV = W$$

$$\Rightarrow \quad = \mathcal{E} I\,dt$$

$$\frac{dW}{dt} = \mathcal{E} I = L I \frac{dI}{dt} + R I^2 \quad \text{—} \; (2)$$

The total workdone by battery at time interval T in which current change from 0 to I_T is

$$W = \int_0^T \mathcal{E} I\,dt = L \int_0^T I \frac{dI}{dt}\,dt + R \int_0^T I^2\,dt$$

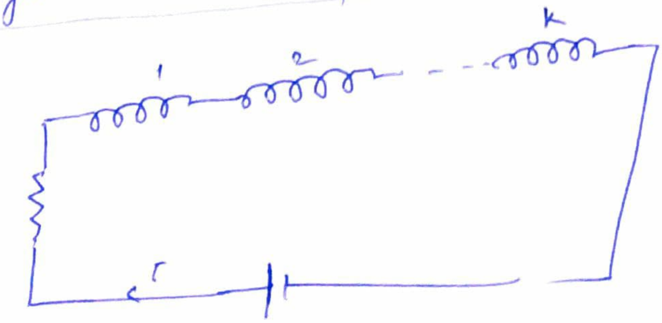
$$= L \int_0^{I_T} I\,dI + R \int_0^T I^2\,dt$$

$$W = \frac{1}{2} L I_T^2 + R \int_0^T \cancel{I}^2\,dt \quad \text{—} \; (3)$$

First term on RHS is energy stored in the inductance in time T and

Second term is energy dissipated as heat in the resistance.

(b) The magnetic energy stored in a series of inductances



Consider the current in all the circuit is initially zero and that attain max. value (constant) at $t = T$.

At ~~only~~ any instant 't' the current in each circuit '$I_k(t)$'

and flux through it $\phi_k(t)$ will be some fraction α of their ultimate values

$$\left.\begin{aligned} I_k(t) &= \alpha I_k \\ \text{and} \quad \phi_k(t) &= \alpha \phi_k \end{aligned}\right\} \quad (4)$$

The induced e.m.f. in k^{th} circuit

$$\mathcal{E}_k = \frac{d\phi_k(t)}{dt} \quad (5)$$

The total work done by circuit k is

$$W_k = \int_0^T \mathcal{E}_k i_k(t)\, dt = I_k \phi_k \int_0^T \alpha \frac{d\alpha}{dt}\, dt$$

$$= I_k \phi_k \int_0^1 \alpha\, d\alpha$$

$$W_k = \frac{1}{2} I_k \phi_k \quad (6)$$

On summing overall the circuit

$$W = \sum_{k=1} \frac{1}{2} I_k \phi_k \quad (7)$$

Since flux depends on the self and mutual inductances

then $$\phi_k = L_k I_k + \sum_{j \neq k} M_{kj} I_j \quad (8) \qquad \{ \phi_2 = M_{21} I_1$$

then

$$W = \frac{1}{2}\sum_k L_k I_k^2 + \frac{1}{2}\sum_k \sum_{\substack{j \\ j \neq k}} M_{kj} I_k I_j \quad \text{— (9)}$$

for pair of coils (two coils)

$$W = \frac{1}{2}L_1 I_1^2 + \frac{1}{2}L_2 I_2^2 + M I_1 I_2 \quad \text{— (10)}$$

where $M_{12} = M_{21} = M$

for any circuit we know that-

$$\phi_k = \oint_k \vec{B}\cdot\vec{ds} = \oint_k (\nabla\times\vec{A})\cdot\vec{ds}$$

$$\phi_k = \oint_k \vec{A}\cdot\vec{dl} \quad \text{— (11)}$$

then eqn (7) becomes

$$W = \frac{1}{2}\sum_k I_k \phi_k = \frac{1}{2}\sum_k I_k \oint_k \vec{A}\cdot\vec{dl} \quad \text{— (12)}$$

for overall space $\Sigma \rightarrow \int$

then

$$W = \frac{1}{2}\iint I_k \vec{A}\cdot\vec{dl} = \frac{1}{2}\iint \vec{A}\cdot I_k\vec{dl}$$

$$= \frac{1}{2}\int_v (\vec{A}\cdot\vec{J})\, d\tau \quad \text{— (13)}$$

we know
$I_k = \vec{J}\cdot\vec{ds}$
$I_k\vec{dl} = \vec{J}\cdot d\tau$

We know

$$\vec{J} = \nabla \times \vec{H}$$

then

$$W = \frac{1}{2}\int_V \{\vec{A}\cdot(\nabla\times\vec{H})\}\,d\tau \quad - (14)$$

From vector ~~quantity~~ Identity

$$\nabla\cdot(\vec{A}\times\vec{H}) = \vec{H}\cdot(\nabla\times\vec{A}) - \vec{A}\cdot(\nabla\times\vec{H})$$

We get from eqn (14)

$$W = \frac{1}{2}\int_V \{\vec{H}\cdot(\nabla\times\vec{A})\}\,d\tau - \frac{1}{2}\int \{\nabla\cdot(\vec{A}\times\vec{H})\}\,d\tau \quad - (15)$$

$$W = \frac{1}{2}\int_V (\vec{H}\cdot\vec{B})\,d\tau - \frac{1}{2}\int_V (\vec{A}\times\vec{H})\cdot d\vec{s} \quad - (16)$$

Since volume integral is to be taken over all space and surface integral must be taken over the sphere at infinity and H varies as r^{-3} and A as r^{-2}

the surface integral vanishes as $r \to \infty$

Hence

$$W = \frac{1}{2}\int_V (\vec{H}\cdot\vec{B})\,d\tau \quad - (17)$$

This indicates that magnetic energy is distributed throughout the region occupied by field with density $\frac{1}{2}(H\cdot B)$

we know $B = \mu H$

them

density is $\frac{1}{2}\mu H^2$ or $\frac{1}{2}\frac{B^2}{\mu}$

Then

$$W = \frac{1}{2}\int_v \frac{B^2}{\mu} d\tau = \frac{1}{2}\mu \int H^2 d\tau$$

Maxwell equations

The following law's specifying the div and curl of E and M fields were

(i) $\nabla \cdot \vec{E} = \rho / \epsilon_0$ Gauss law in electrostatics — (1)

$\nabla \cdot D = \rho$

(ii) $\nabla \cdot \vec{B} = 0$ no name or Gauss law in magnetostatic — (2)

(iii) $\nabla \times \vec{E} = -\frac{\partial B}{\partial t}$ Faradays law of induction — (3)

(iv) $\nabla \times \vec{B} = \mu_0 \vec{J}$ Ampere law — (4)

$\nabla \times \vec{H} = \vec{J}$

The first three are general equation and are valid for static and dynamic fields.

The ~~forth~~ fourth eqn was derived from steady-state observation and its validity have to examine for time varying fields.

Taking div. of eqn (4), we get

$\nabla \cdot (\nabla \times H) = \nabla \cdot \vec{J} = 0$ — (5) { div of curl is zero

This is true for steady state phenomenon. $\{ \frac{\partial \rho}{\partial t} = 0 \}$

But for time varying current, From the law of conservation of charge, the continuity eqn is

$$\nabla \cdot J + \frac{\partial \rho}{\partial t} = 0 \quad \text{— (6)}$$

Therefore the left side of eqn (5) is zero but right side is not zero (from eqn (6))

Maxwell appreciated this situation and suggested a way out.

He realised that the difficulty arose from the ~~incompatible~~ incomplete definition of the total current density in eqn (4)

using eqn (1), we can write eqn (6) as

$$\nabla \cdot \vec{J} = -\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial t}(\nabla \cdot D) = \nabla \cdot \left(-\frac{\partial D}{\partial t}\right)$$

$$\nabla \cdot \left(\vec{J} + \frac{\partial D}{\partial t}\right) = 0 \quad \text{— (7)}$$

then maxwell replaced $\vec{J}$ in Ampere's law by $\vec{J} + \frac{\partial D}{\partial t}$.

The maxwell modified the Ampere's law as

$$\nabla \cdot \vec{H} = \vec{J} + \frac{\partial D}{\partial t} \quad \text{— (8)}$$

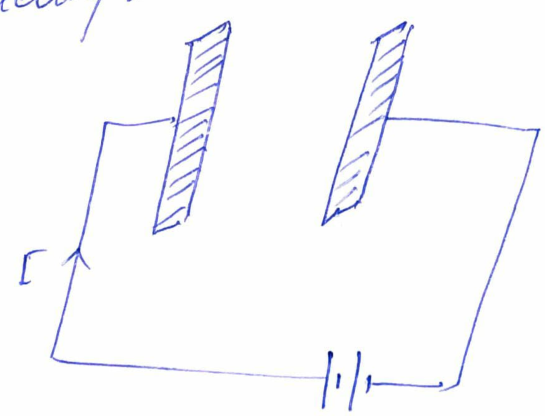
This is valid for steady state and time varying current. (time dependent field)

$\vec{J}$ is called "conduction current density."

The second term $\frac{\partial D}{\partial t}$ is arises from the variation of electric displacement with time is called "displacement current density"

The displacement current does not have the significance of current in sense of being motion of a ~~current in~~ charges.

Let us consider the capacitor connected with battery through the conducting wire



The current flowing in the ~~capacitor~~ conducting wire is equal to the rate of change of charge on the plates

$$I = \frac{dq}{dt} \quad - (9)$$

The q is charge on plate of capacitor, which is related to field in capacitor.

$$\vec{E} = \frac{\sigma}{\epsilon_0} = \frac{Q}{\epsilon_0 A} \quad \text{— (10)}$$

A is area of plate

$$I = \frac{dQ}{dt} = \epsilon_0 A \frac{\partial \vec{E}}{dt} = A \frac{\partial D}{\partial t} \qquad (\epsilon_0 E = D)$$

i.e.

$$\frac{I}{A} = \frac{\partial D}{\partial t}$$

$$J = \frac{\partial D}{\partial t} \quad \text{— (11)}$$

Therefore quantity $\frac{\partial D}{\partial t}$ is current density.

It is interpreted as ~~[illegible]~~ density of some current ~~which~~ corresponds to current, must flow in the space even in vacuum b/w the pair of plates when charge plates are connected with wire, thus completing the conduction current.

Again consider a copper wire in which ~~and~~ two types of current can be obtained.

Let electric field in coper wire is

$$\vec{E} = E_0 e^{-j\omega t}$$

then $\vec{J} = \sigma \vec{E} = \sigma E_0 e^{-j\omega t}$

then

$$|J|^2 = \sigma^2 |E_0|^2$$

$$\vec{D} = \epsilon_0 E_0 e^{-j\omega t}$$

$$\frac{\partial \vec{D}}{\partial t} = j\omega \epsilon_0 E_0 \, e^{-j\omega t}$$

Hence

$$\left|\frac{\partial \vec{D}}{\partial t}\right|^2 = \omega^2 \epsilon_0^2 |E_0|^2$$

then

$$\left|\frac{\vec{J}}{\partial \vec{D}/\partial t}\right| = \frac{\sigma}{\omega \epsilon_0} \quad — (12)$$

for copper $\sigma = 5.9 \times 10^7$ s/m

then $\frac{\sigma}{\omega \epsilon_0} = \frac{10^{19}}{\omega}$

thus the ratio is very large for all freq.

therefore displacement current are not significant as current due to the motion of free charge in the circuit.

the fourth eq^n, which the field vector $\vec{E}, \vec{D}, \vec{B}, \vec{H}$ satisfy everywhere are

$$\left.\begin{array}{ll} (i) & \nabla \cdot \vec{D} = \rho \\ (ii) & \nabla \cdot B = 0 \\ (iii) & \nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0 \\ (iv) & \nabla \times H = \vec{J} + \frac{\partial D}{\partial t} \end{array}\right] \quad — (13)$$

these eq^n are fundamental eq^n of e.m. field. and are known as Maxwell's eq^n.

Decay of free charge

One of important deductions that can be made from maxwell's eq^n is related to decays of free charge.

from (iv) eq^n

$$\nabla \times H = \vec{J} + \frac{\partial \vec{D}}{\partial t} = \sigma \vec{E} + \epsilon \frac{\partial \vec{E}}{\partial t} \quad — (1)$$

taking div. of eq^n (1)

$$\nabla \cdot (\nabla \times \vec{H}) = \sigma \nabla \cdot \vec{E} + \epsilon \frac{\partial}{\partial t} (\nabla \cdot E)$$

$$= \frac{\sigma \rho}{\epsilon} + \frac{\partial \rho}{\partial t} = 0 \qquad \{\text{div. of curl} = 0$$

$$\frac{\partial \rho}{\partial t} = -\sigma \rho / \epsilon$$

$$\frac{\partial \rho}{\rho} = -\frac{\sigma}{\epsilon} \partial t \quad — (2)$$

on integrating, we get

$$\left. \begin{aligned} \rho &= \rho_0 e^{-t/\tau} \\ \text{where } \tau &= \epsilon / \sigma \end{aligned} \right\} — (3)$$

τ is relaxation time

eq^n (3) shows that any original distribution of charge decays exponentially at a rate, which is independent of any other e.m. ~~distribution~~ disturbances.

for copper $\tau = \frac{\epsilon_0}{\sigma} = \frac{8.85 \times 10^{-12}}{5.8 \times 10^{7}} = 1.5 \times 10^{-19}$ sec.

Maxwell eqn in vaccum or free space

for free space $\rho = 0$ and $\vec{J} = 0$
the maxwell eqn

$$\nabla \cdot \vec{E} = \rho/\epsilon_0 \qquad \nabla \cdot \vec{B} = 0$$

$$\nabla \cdot \vec{E} = 0$$

$$\nabla \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

$$\nabla \times \vec{E} = -\frac{\partial B}{\partial t} \qquad \nabla \times B = \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} \qquad - (1)$$

Maxwell eqn in matter/media

When electric and magnetic field are existed in matter then it is polarized.

for inside polarized matter, there will be accumulation of bound charge and current over which it is exerted no control.

therefore in matter there are two type of charges

(i) free charge (current canbe controlled

(ii) bound charge (no control on current)

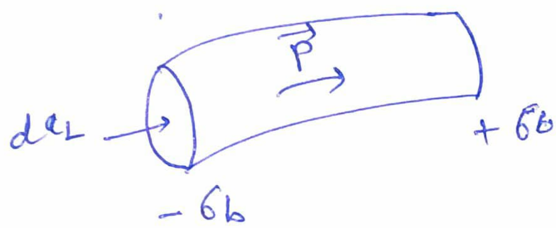
The electric polarized produced a bound charge density

$$\rho_b = -\nabla \cdot \vec{P} \quad \text{—— (1)}$$

$\vec{P} \rightarrow$ polarization vector

Similarly magnetic polarization (or magnetization) M results in a bound current

$$\vec{J}_b = \nabla \times \vec{M} \quad \text{—— (2)}$$



Let change in electric polarization produces the flow of bound charge (J_p)

~~then total current density~~

$$\cancel{J = J_p + J_f} \quad - (3)$$

~~due to bound charge~~ ~~due to free charge~~

Let us consider a tiny chunk of polarized material. The polarization introduces a charge density $\sigma_p = P$ at one end $-\sigma_b$ at the other.

If P now increases a bit, the charge on each end increases accordingly.

Then net current

$$dI = \frac{\partial \sigma_b}{\partial t} da_{\perp} = \frac{\partial \vec{P}}{\partial t} d\sigma_{\perp}$$

The current density

$$J_p = \frac{\partial \vec{P}}{\partial t} \quad — \quad (4)$$

This polarization current has nothing whatever to do with bound current.

Latter is ~~involved~~ associating with magnetization of material and involves the spin and orbital motion of electrons.

$J_p \rightarrow$ by linear motion of charge when electric polarization changes.

If $\vec{P}$ points to the right and is increasing, then each plus charge move a bit to right and each minus charge to the left.

The cumulative effect is polarization current J_p

eq^n (4) is consistent with continuity eq^n

$$\nabla \cdot J_p = \nabla \cdot \frac{\partial \vec{P}}{\partial t}$$

$$= \frac{\partial}{\partial t}(\nabla \cdot P) = -\frac{\partial \rho_b}{\partial t}$$

$$\nabla \cdot J_p = -\frac{\partial \rho_b}{\partial t} \quad — (5)$$

the continuity eqn is satisfied.

Therefore bound charge is also conserved.

The total charge density is separated into two parts

$$\rho = \underbrace{\rho_f}_{\text{free charge}} + \underbrace{\rho_b}_{\text{bound charge}}$$

$$\rho = \rho_f - \nabla \cdot \vec{P} \quad — (6)$$

The total current density into three parts

$$\vec{J} = \underbrace{J_f}_{\text{free charge}} + \underbrace{J_b}_{\text{bound current}} + \underbrace{J_p}_{\text{bound charge}} \quad — (7)$$

From Gauss's law

$$\nabla \cdot E = \frac{1}{\epsilon_0}(\rho_f + \nabla \cdot P)$$

or

$$\nabla \cdot \vec{D} = \rho_f \quad \text{—— (8)}$$

where $\vec{D}$ is given as in static case

$$\vec{D} = \epsilon_0 \vec{E} + \vec{P} \quad \text{—— (9)}$$

From Ampere's law with maxwell term

$$\nabla \times \vec{B} = \mu_0 \left(J_f + \nabla \times M + \frac{\partial \vec{P}}{\partial t} \right) + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

$$\nabla \times \vec{H} = J_f + \frac{\partial D}{\partial t} \quad \text{—— (10)}$$

where $$\vec{H} = \frac{\vec{B}}{\mu_0} - \vec{M} \quad \text{—— (11)}$$

Faraday's law and $\nabla \cdot B$ are not affected by separation of charge and current since they do not involve ρ and $\vec{J}$

In terms of free charge maxwell eq^n are

(i) $\nabla \cdot \vec{D} = \rho_f$ (iii) $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$

(ii) $\nabla \cdot \vec{B} = 0$ (iv) $\nabla \times \vec{H} = J_f + \frac{\partial \vec{D}}{\partial t}$ } — (12)

the generalised form of maxwell eq^n.
Some time, people regard these as "true" Maxwell eq^n

these eq^n's reflect the convenient division of charge and current into free and non-free part.

They have the disadvantages of hybrid notation since they contain both $\vec{E}$ and $\vec{D}$, both $\vec{B}$ and $\vec{H}$.

They depends on nature of material

(i) for linear media

$$\vec{P} = \epsilon_0 \chi_e \vec{E} \text{ and } M = \chi_M H \quad — (13)$$

so

$$D = \epsilon E \quad \text{and} \quad H = \frac{B}{\mu} \quad — (14)$$

where $\quad \epsilon = \epsilon_0 (1 + \chi_e)$

and $\quad \mu = \mu_0 (1 + \chi_M)$

D is called the electric displacement

so second term in Ampere's eq^n law / Maxwell eq^n (iv) is called the displacement current.

from eq^n (12) Maxwell eq^n for free space

for free space $\vec{P} = 0$ $\qquad \vec{M} = 0$

then $\vec{D} = \epsilon_0 \vec{E} + \vec{P}$ $\qquad$ and $\quad \vec{H} = \frac{\vec{B}}{\mu_0} - \vec{M}$

$D = \epsilon_0 E$ $\qquad H = \frac{B}{\mu_0}$

Then (i) $\nabla \cdot E = \rho/\epsilon_0$ $\qquad$ (ii) $\nabla \cdot \vec{B} = 0$

[for free space $\rho = 0$] $\qquad$ (iii) $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$

$\nabla \cdot E = 0$ $\qquad$ (iv) $\nabla \times \vec{B} = \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$ $\quad$ {for free space, $J = 0$

Boundary Conditions

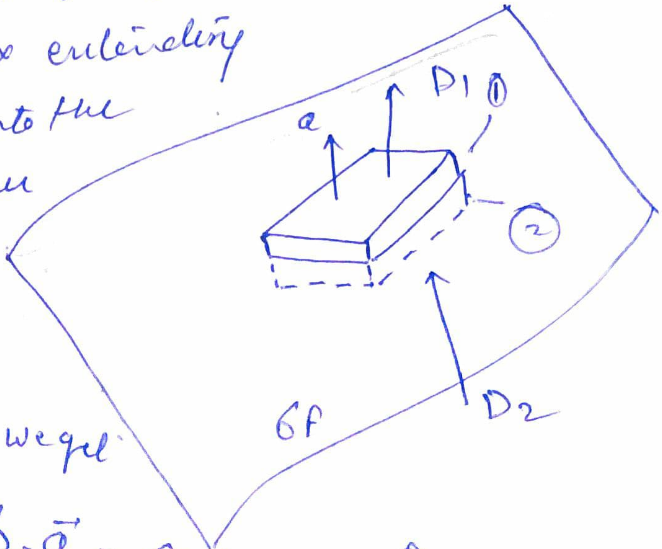
The fields $\vec{E}, \vec{B}, \vec{D}$ and $\vec{H}$ will be discontinuous at a boundary b/w two different media or at a surface that carries charge density σ or current density $\vec{K}$

We start from Maxwell's eqⁿ in integral form

(i) $\oint_s \vec{D}\cdot d\vec{a} = Q_{f\,enc}$
(ii) $\oint_s \vec{B}\cdot d\vec{a} = 0$
} over any closed surface S

(iii) $\oint_C \vec{E}\cdot d\vec{l} = -\frac{d}{dt}\int_s \vec{B}\cdot d\vec{a}$
(iv) $\oint_C \vec{H}\cdot d\vec{l} = I_{f\,enc} + \frac{d}{dt}\int_s \vec{D}\cdot d\vec{a}$
} for any surface S bounded by the closed loop C

Consider a tiny, ~~surf~~ wafer thin Gaussian pillbox extending just slightly into the material on either side of boundary

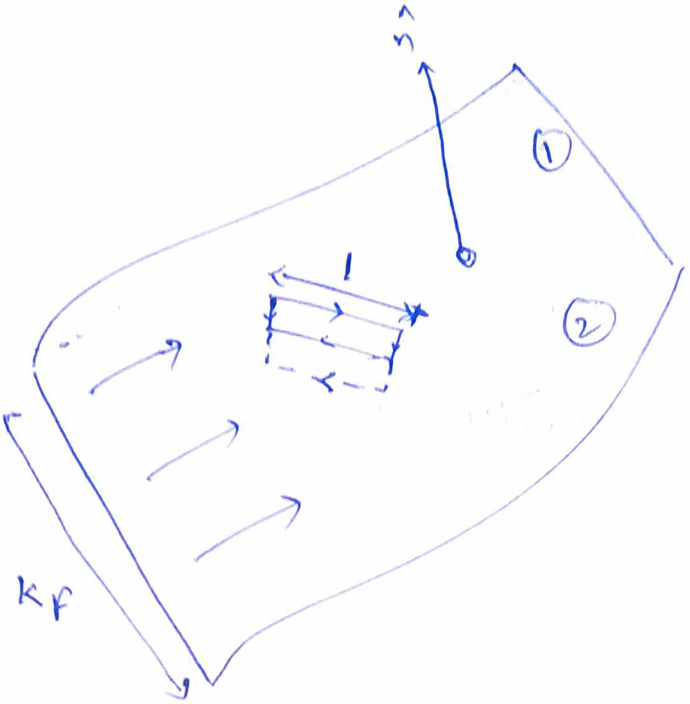


Applying (i), we get

$$\vec{D}_1\cdot\vec{a} - \vec{D}_2\cdot\vec{a} = \sigma_f\, a \qquad — ①$$

Thus component of $\vec{D}$ that is perpendicular to interface is discontinuous in amount

$D_1^{\perp} - D_2^{\perp} = \sigma_f$ —— (2)



Applying (ii) ~~for above diagram~~, we get

$$B_1^{\perp} - B_2^{\perp} = 0 \quad \text{—— (2)}$$

Let a very thin Amperian loop straddling the surface of above figure.

Applying (iii) on it

$$\vec{E}_1 \cdot l - E_2 \cdot l = -\frac{d}{dt}\int_S \vec{B} \cdot d\vec{a} \quad - (3)$$

But in the limit as the width of loop goes to zero the flux vanishes

then

$$E_1^{\parallel} - E_2^{\parallel} = 0 \quad \text{—— (4)}$$

that is component of $\vec{E}$ parallel to interface are continuous across the boundary

Similarly applying (iv) we get

$$H_1 \cdot l - H_2 \cdot l = I_{fenc} \quad -(5)$$

where I_{fenc} is free current passing through the loop

No volume current density will contribute (infinitesimal width) but surface current can.

If $\hat{n}$ is a unit vector perpendicular to the interface so that $\hat{n} \times l$ is normal to the loop then

$$I_{fenc} = k_f \cdot (\hat{n} \times l) = (k_f \times \hat{n}) \cdot l$$

hence

$$H_1^{\parallel} - H_2^{\parallel} = k_f \times \hat{n} \quad -(6)$$

So the parallel component of H are discontinuous by amount proportional to the free surface current density.

The general boundary conditions for electrodynamics for linear media

(i) $\epsilon_1 E_1^{\perp} - \epsilon_2 E_2^{\perp} = \sigma_f$ (iii) $E_1^{\parallel} - E_2^{\parallel} = 0$

(ii) $B_1^{\perp} - B_2^{\perp} = 0$ (iv) $\frac{1}{\mu_1} B_1^{\parallel} - \frac{1}{\mu_2} B_2^{\parallel} = \vec{K}_f \times \hat{n}$

(7)

If there is no free charge or free current at interface then

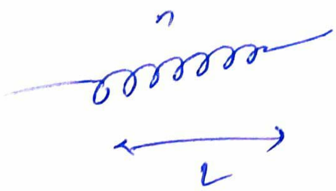
(i) $\epsilon_1 E_1^{\perp} - \epsilon_2 E_2^{\perp} = 0$ (iii) $E_1^{\parallel} - E_2^{\parallel} = 0$

(ii) $B_1^{\perp} - B_2^{\perp} = 0$ (iv) $\frac{1}{\mu_1} B_1^{\parallel} - \frac{1}{\mu_2} B_2^{\parallel} = 0$

8

Self inductance in a solenoid

Let us consider a solenoid having N turns in it and cross section area is A



then self inductance

$$L = \frac{\phi_T}{i}$$

where $\phi_T = N\phi_B$

then $L = \dfrac{N\phi_B}{i}$

where i is current in solenoid, ϕ_T is total flux

~~let is length of solenoid and~~ l is per unit length of solenoid

then

$$\phi_T = N\phi_B = N(B \cdot A)$$

The number of flux linkage in the length of L is

$$N\phi_B = (nl(BA)$$

$n \rightarrow$ no. of turns per unit length $\left(\frac{N}{L}\right)$

B is magnetic induction

B for a solenoid is

$$B = \mu_0 n i$$

$$N\phi_B = \mu_0 n^2 l i A$$

$$L = \frac{N\phi_B}{l} = \mu_0 n^2 l A$$

$L \propto Al$ (volume of solenoid)

$L \propto n^2$ (square of no. of turns per unit length)

Example

① Calculate the inductance of a solenoid of 2000 turns wound uniformly over a length of 50 cm on a cylindrical paper tube 4cm in diameter. The medium is air

Solution

from relation

L is solenoid

$L = \mu_0 n^2 l A$ when $N = nl$

$N = 2000$ turns

$\mu_0 = 4\pi \times 10^{-7}$ Weber/Amp m

$A = \pi r^2 = \pi (2 \times 10^{-2})$ m

$$L = \mu_0 \frac{N^2}{l} A$$

$$= \frac{4\pi \times 10^{-7} \times 2000 \times 2000 \times \pi (2 \times 10^{-2})^2}{0.5}$$

$\times 10$

$$= \frac{4\pi \times 4 \times 4\pi \times 10^{-4}}{5}$$

$= 12.6 \times 10^{-3}$ henrage

$= 12.6$ MHz

Example

(2) A coil has an inductance of 5.0 henry and resistance of 20 ohms. If a 100 volt e.m.f. is applied, calculate the energy stored in the magnetic field after the current has built up to its maximum value.

Solution Maximum current is

$$I = \frac{E}{R} = \frac{100}{20} = 5.0 \text{ Amp}$$

The energy store in inductor is

$$U_B = \frac{1}{2} L I^2$$

$$= \frac{1}{2} \times 5 \times (5)^2$$

$$= \frac{125}{2} = 62.5 \text{ Joule}.$$

Example

(3) A straight solenoid has 50 turns per cm in the primary coil and 200 turns in secondary coil. The area of cross-section of solenoid is $4 \, cm^2$. Calculate the mutual inductance

Solution from relation of mutual inductance

$$M = \mu_0 N_p N_s A_p$$

given $N_p = 50$ turns per length (cm)

$= 50 \times 100$ turns per metre

$= 5000$

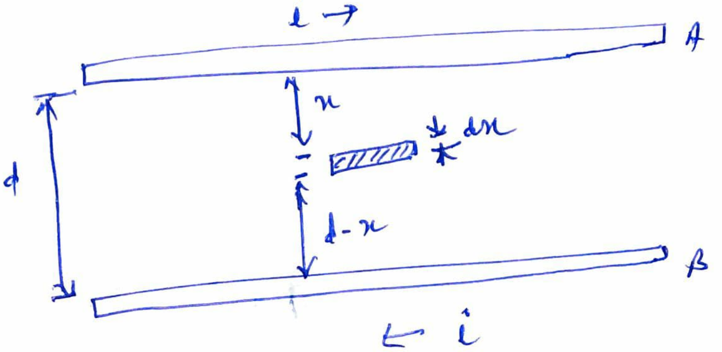
then

$M = 4\pi \times 10^{-7} \times 5000 \times 200 \times 4 \times 10^{-4}$ henry

$= 50.27 \times 10^{-5}$ henry

Example

(4) Calculate the self-inductance of two parallel wires



Solution

Consider a circuit of two parallel wire A and B separated by distance d

the current is going through A and returning by B

let radius of each wire is r

the field due to current in two wires

$$B = \frac{\mu_0 i}{2\pi x} - \frac{\mu_0 i}{2\pi (d-x)} \qquad \left\{ B = \frac{\mu_0 i}{2\pi r} \right.$$

the flux through dotted area (unit length & width is dx)

$$d\phi_B = B\,dx$$

then total flux $\phi_B = \int_0^d d\phi_B = \int_r^{d-r} \left(\frac{1}{x} + \frac{1}{d-x}\right) \frac{\mu_0}{2\pi}\, dx$

$$\phi_B = \frac{\mu_0 i}{2\pi} \int_r^{d-r} \left(\frac{1}{x} + \frac{1}{d-x} \right) dx$$

$$= \frac{\mu_0 i}{2\pi} \Big[\log x - \log (d-x) \Big]_r^{d-r}$$

$$= \frac{\mu_0 i}{2\pi} \left[\log \frac{x}{d-x} \right]_r^{d-r}$$

$$= \frac{\mu_0 i}{2\pi} \left[\log \frac{d-r}{r} - \log \frac{r}{d-r} \right]$$

$$= \frac{\mu_0 i}{2\pi} \left[2 \log \left(\frac{d-r}{r} \right) \right]$$

Now when i varies the induced e.m.f according to faraday's law

$$\varepsilon = \frac{-d\phi_B}{dt}$$

$$\cancel{= - \frac{2\mu_0}{2\pi} \left[\log \frac{d-r}{r} \right]}$$

$$\varepsilon = - \frac{2\mu_0}{2\pi} \log_e \left(\frac{d-r}{r} \right) \cdot \frac{di}{dt}$$

$$= L \frac{di}{dt}$$

Then

$$L = \frac{\mu_0}{\pi} \log \frac{d-r}{r}$$

$$= \frac{\mu_0}{\pi} \times 2.303 \log_{10} \frac{d-r}{r}$$

Example 4

An all metal aeroplane drives down vertically at 300 km/sec at a plane where the horizontal component of earth's field is 0.4 oersted. If the wing span is 30 meters, what will be the resulting potential difference b/w the tips?

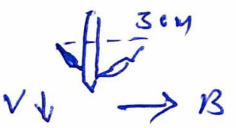
Solution: vertical distance covered in one second

$300\ km = 300 \times 10^3\ m = 3 \times 10^5\ m$

the distance b/w tips of wing = 30 m

the area swept by wing span in one second

$$\frac{dA}{dt} = 3 \times 10^5 \times 30\ m^2/sec$$



$$= 9 \times 10^6\ m^2/sec$$

The potential difference induced b/w the wing tips

$$\varepsilon = -\frac{d\phi}{dt} = -\frac{d}{dt}(\vec{B}\cdot\vec{A}) = -\frac{d}{dt}(HA)$$

$$= H\frac{dA}{dt}$$

$$= 0.4 \times 10^{-4}\ weber/m^2 \times 9 \times 10^5\ m^2/sec$$

$$= 360\ volt$$

$\{$ 1 weber/m² = 10^4 gauss or (oersted)