Unit-V

(A) Electromagnetic potential

Magnetic scalar and vector potential

We know that electrostatic field is simply related with electric potential V as

$$\vec{E} = -\nabla V \quad \text{--- (1)}$$

Similarly we can define a potential associated with magnetostatic field $\vec{B}$

The magnetic potential could be scalar V_M or vector $\vec{A}$

To define V_M and $\vec{A}$ involves two important identities

$$\nabla \times (\nabla V) = 0 \quad \text{--- (2)}$$

$$\nabla \cdot (\nabla \times \vec{A}) = 0 \quad \text{--- (3)}$$

which must always hold for any scalar field V and vector field $\vec{A}$.

Just as $E = -\nabla V$, we define the magnetic scalar potential V_M (in Ampere) as related to H according to

$$\vec{H} = -\nabla V_M \quad \text{if } \vec{J} = 0 \quad \text{--- (4)}$$

The condition attached to this eqn ($\vec{J}=0$) is important

We know that

$$J = \nabla \times H = \nabla \times (-\nabla V_M) = 0 \quad \text{—} \quad (5)$$

Since V_M must satisfy the condition in eqn (2)

Therefore the V_M is only defined in the region where $\vec{J}=0$ as in eqn (4)

Also noted that V_M satisfies laplace eqn just as V does for electrostatic fields

$$\nabla^2 V_M = 0 \quad (J=0) \quad \text{—} \quad (6)$$

for magnetostatic field

$$\nabla \cdot \vec{B} = 0 \quad (\text{as in eqn maxwell eqn})$$

To satisfy eqn above eqn, we define the vector magnetic potential $\vec{A}$ (in Wb/m) such that

$$\vec{B} = \nabla \times \vec{A} \quad \text{—} \quad (7)$$

Just as we define

for electrostatic field

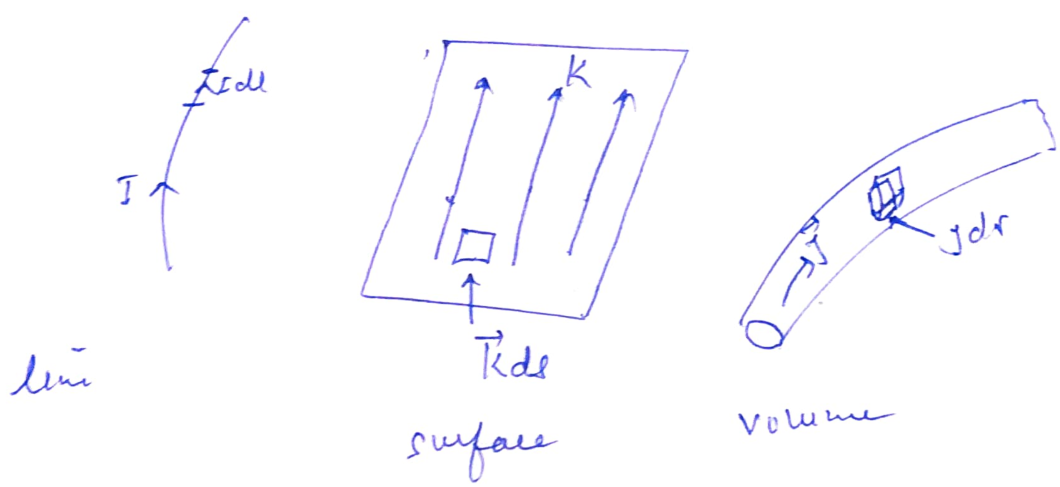
$$V = \int \frac{dQ}{4\pi\epsilon_0 r} \quad -(8)$$

we can define

$$\vec{A} = \int_L \frac{\mu_0}{4\pi} \frac{I\,\vec{dl}}{R} \quad \text{for line current} \quad -(9)$$

$$\vec{A} = \int_S \frac{\mu_0}{4\pi} \frac{\vec{K}\,ds}{R} \quad \text{for surface current} \quad -(10)$$

$$\vec{A} = \int \frac{\mu_0 J\,dv}{4\pi R} \quad \text{for volume current} \quad -(11)$$

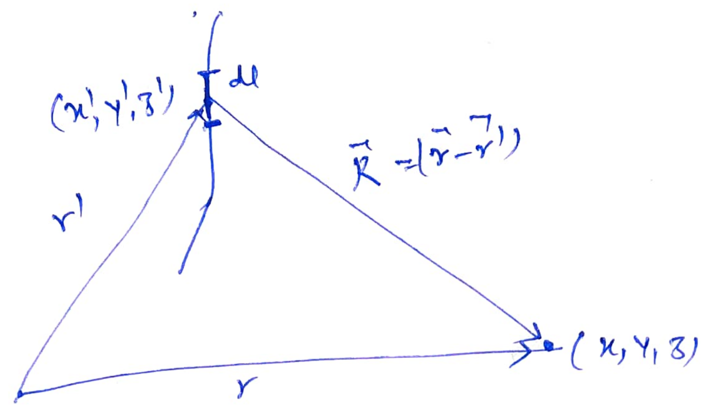


Rather than obtaining eqn (9)-(11) from eqn (8), an alternative approach would be to obtain eqn (9) to (11) from Biot-savart law as given as

$$\vec{B} = \frac{\mu_0}{4\pi} \int_L \frac{I\,\vec{dl}' \times \vec{R}}{R^3} \quad -(12)$$

where $\vec{R}$ is distance vector from line element $d\vec{l}'$ at the source point (x', y', z') to the field point (x, y, z) as shown in fig

and $|\vec{R}| = R$



$$R = |\vec{r} - \vec{r}'| = \left[(x-x')^2 + (y-y')^2 + (z-z')^2\right]^{1/2} \quad \text{— (13)}$$

then

$$\nabla\left(\frac{1}{R}\right) = -\frac{(x-x')\hat{a}_x + (y-y')\hat{a}_y + (z-z')\hat{a}_z}{\left[(x-x')^2 + (y-y')^2 + (z-z')^2\right]^{3/2}} = -\frac{\vec{R}}{R^3}$$

or

$$\frac{\vec{R}}{R^3} = -\nabla\left(\frac{1}{R}\right) = \left(\frac{\hat{a}_R}{R^2}\right) \quad \text{— (14)}$$

$\left\{ \begin{array}{l} \vec{R} = R\hat{a}_R \\ \hat{a}_R \rightarrow \text{unit vector} \end{array} \right.$

using eqn (14) in eqn (12), we get

$$\vec{B} = \frac{\mu_0}{4\pi} \int_2 I d\vec{l}' \times \nabla\left(\frac{1}{R}\right) \quad \text{— (15)}$$

Applying vector identity

$$\nabla \times (f\vec{F}) = f\,\nabla \times \vec{F} + (\nabla f) \times \vec{F} \quad \text{— (16)}$$

where f is scalar field

$\vec{F}$ is vector field

Taking $f = \frac{1}{R}$ and $F = \vec{dl'}$, we get

$$\vec{dl'} \times \nabla\left(\frac{1}{R}\right) = \frac{1}{R} \nabla \times \vec{dl'} - \nabla \times \left(\frac{\vec{dl'}}{R}\right)$$

Since ∇ operates with ~~on~~ respect to (x, y, z) while $\vec{dl'}$ is function of (x', y', z')

$$\nabla \times \vec{dl'} = 0$$

Hence $$\vec{dl'} \times \nabla\left(\frac{1}{R}\right) = - \nabla \times \left(\frac{\vec{dl'}}{R}\right) \quad \text{— (17)}$$

using eqn (17) in eqn (15), we get

$$\vec{B} = \nabla \times \int_L \frac{\mu_0 I \vec{dl'}}{4\pi R} \quad \text{— (18)}$$

Comparing eqn (7) $\{ \vec{B} = \nabla \times \vec{A} \}$ and eqn (18), we get

$$\vec{A} = \int_L \frac{\mu_0 I}{4\pi} \frac{\vec{dl'}}{R} \quad \text{— (19)}$$

verified eqn (9)

we know that magnetic flux in terms of $\vec{B}$ is

$$\phi_B = \int_S \vec{B} \cdot \vec{ds} \quad \text{— (20)}$$

Using eqn 7 we get

$$\phi_B = \int_S (\nabla \times \vec{A}) \cdot d\vec{s}$$

Now applying stoke's curl theorem

$$\phi = \oint_L \vec{A} \cdot d\vec{l} \quad - (21)$$

Thus the magnetic flux through a given area can be found by using eqn (20) or (21).

Also magnetic field can be determined using either V_M or $\vec{A}$

The V_M can be used only in a source free region.

The use of $\vec{A}$ provides a powerful elegant approach to solving E.M. problem.

Electromagnetic potential

The complete description of an e.m. field can be obtained by solving maxwell's eq^n's. The process becomes simple if the eq^n are written in suitable form. The reduction of the no. of eq^n's is convenient by introducing new quantities called "electromagnetic potential"

The electrostatic field in terms of scalar potential

$$E = -\nabla\phi \quad -(1)$$

and magnetic field in terms of vector potential

$$\vec{B} = \nabla\times\vec{A} \quad -(2)$$

Now consider potential in e.m. field when $\vec{E}$ and $\vec{B}$ fields are time varying.

The field vectors satisfy the maxwell's eq^n at every point

$$\left.\begin{aligned} \operatorname{div}\vec{D} &= \rho \\ \operatorname{div}\vec{B} &= 0 \\ \operatorname{curl}\vec{H} &= \vec{J} + \frac{\partial \vec{D}}{\partial t} \\ \operatorname{Curl}\vec{E} &= -\frac{\partial \vec{B}}{\partial t} \end{aligned}\right\} \quad -(3)$$

from eqⁿ (iv) of ③

$$\nabla\times\vec{E} = -\frac{\partial}{\partial t}(\text{curl }\vec{A}) = -\frac{\partial}{\partial t}(\nabla\times\vec{A})$$

$$\nabla\times\left(\vec{E}+\frac{\partial A}{\partial t}\right) = 0 \quad -④$$

The quantity i.e. field vector $\vec{E}+\frac{\partial A}{\partial t}$ is irrotational and must be equal to gradient of some scalar function. we write

$$\vec{E}+\frac{\partial\vec{A}}{\partial t} = -\text{grad }\phi$$

$$E = -\text{grad }\phi + \frac{\partial A}{\partial t} \quad -⑤$$

There a vector $\vec{A}$ and a scalar ϕ both being function of position and time. these are called e.m. potential.

scalar ϕ is called the scalar potential.

vector $\vec{A}$ is called vector potential.

from eq(iii) of ③,

$$\mu \text{ curl }\vec{H} = \mu\vec{J} + \mu\frac{\partial\vec{D}}{\partial t}$$

$$\text{curl }\vec{B} = \mu\vec{J} + \mu\epsilon\frac{\partial\vec{E}}{\partial t} \quad -⑥$$

$$\text{curl}(\text{curl }\vec{A}) = \mu\vec{J} + \mu\epsilon\frac{\partial}{\partial t}\left(-\text{grad }\phi - \frac{\partial A}{\partial t}\right)$$

using vector identity

$$\nabla \times (\nabla \times \vec{A}) = \nabla(\nabla \cdot \vec{A}) - (\nabla \cdot \nabla)\vec{A}$$

we get

$$\nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A} = \mu J + \mu\epsilon \frac{\partial}{\partial t}(\nabla\phi) - \mu\epsilon \frac{\partial^2 \vec{A}}{\partial t}$$

$$\nabla^2 \vec{A} - \mu\epsilon \frac{\partial^2 \vec{A}}{\partial t^2} - \nabla\left(\nabla \cdot \vec{A} + \mu\epsilon \frac{\partial \phi}{\partial t}\right) = -\mu J \quad — (7)$$

Similarly from eqn (i) of (1), we get

$$\nabla \cdot \vec{D} = \rho$$

$$\epsilon \nabla \cdot \vec{E} = \rho$$

$$\nabla \cdot \left(-\nabla\phi - \frac{\partial A}{\partial t}\right) = \rho/\epsilon$$

$$\nabla^2 \phi + \frac{\partial}{\partial t}(\nabla \cdot \vec{A}) = -\rho/\epsilon.$$

$$\nabla^2 \phi - \mu\epsilon \frac{\partial^2 \phi}{\partial t^2} + \frac{\partial}{\partial t}\left(\nabla \cdot \vec{A} + \mu\epsilon \frac{\partial \phi}{\partial t}\right) = -\rho/\epsilon \quad — (8)$$

The eqn (7) and (8) are field equations in term of electromagnetic potential. The Maxwell's eqn's are reduced from four to two by e.m. potential.

Applying Coulomb gauge $\nabla \cdot A = 0$ in eqn (7) and (8)

we get

$$\nabla^2 \vec{A} - \mu\epsilon \frac{\partial^2 \vec{A}}{\partial t^2} - \nabla\left(\mu\epsilon \frac{\partial \phi}{\partial t}\right) = -\mu \vec{J} \quad - (9)$$

and $\nabla^2 \phi = -\rho/\epsilon \quad - (10)$

for steady state field

$$\left.\begin{aligned} \nabla^2 \vec{A} &= -\mu \vec{J} \\ \text{and} \quad \nabla^2 \phi &= -\rho/\epsilon \end{aligned}\right\} \quad - (11)$$

Eqn (11) is known as poisson eqn for vector potential in terms of current density and for scalar potential in terms of charge density.

Non-uniqueness of e.m. potential and concept of Gauge

$\vec{E}$ and $\vec{B}$ in terms of $\vec{A}$ and ϕ are given

$$\vec{E} = -\nabla\phi - \frac{\partial \vec{A}}{\partial t} \quad - (1)$$

and $\vec{B} = \nabla \times \vec{A} \quad - (2)$

From eqn (1) and (2) for given values of $\vec{A}$ and ϕ, $\vec{B}$ and $\vec{E}$ each have only one value i.e. $\vec{A}$ and ϕ determine $\vec{B}$ and $\vec{E}$ uniquely.

But converse is not true i.e. field vector $\vec{B}$ & $\vec{E}$ do not determine $\vec{A}$ and ϕ completely.

Therefore for a given $\vec{A}$ and ϕ, there will be only one $\vec{E}$ and $\vec{B}$ (or $\vec{H}$), while for a given $\vec{E}$ and $\vec{H}$, there can be a infinite of $\vec{A}$ and ϕ's.

This is because the curl of gradient of any scalar vanishes and hence we may add of to $\vec{A}$ the gradient of a scalar Λ without affecting $\vec{B}$

that is $\vec{A}$ is replaced by

$$\vec{A}' = \vec{A} + \nabla\Lambda \quad — (3)$$

eqn (1) becomes

$$\vec{E} = -\nabla\phi - \frac{\partial}{\partial t}(A' - \nabla\Lambda)$$

$$= -\nabla\phi - \frac{\partial A'}{\partial t} + \frac{\partial}{\partial t}(\nabla\Lambda)$$

$$= -\nabla\left(\phi - \frac{\partial \Lambda}{\partial t}\right) - \frac{\partial A'}{\partial t}$$

Then also ϕ must be replaced by

$$\phi' = \phi - \frac{\partial \Lambda}{\partial t} \quad — (4)$$

$\left\{ \text{if we make transformation given by (3)} \right.$

then $\vec{E}$ and $\vec{H}$ remain unchanged under the transformation (3) and (4) i.e.

$$\vec{B} = \text{Curl } \vec{A} = \text{Curl } (\vec{A}' - \nabla\Lambda)$$

$$= \text{Curl } \vec{A}' \quad — (5)$$

and
$$\vec{E} = -\nabla\phi - \frac{\partial \vec{A}}{\partial t}$$

$$= -\nabla\left(\phi' + \frac{\partial\Lambda}{\partial t}\right) - \frac{\partial}{\partial t}(A' - \nabla\Lambda)$$

$$= -\nabla\phi' - \frac{\partial \vec{A}'}{\partial t} \quad — (6)$$

i.e. field vectors remains unchanged for the set (A, ϕ) or (A', ϕ')

So e.m. potential defines field vector uniquely though they themselves are non-unique.

The transformation

$$\left.\begin{aligned} A' &= A + \nabla\Lambda \\ \text{and } \phi' &= \phi - \frac{\partial\Lambda}{\partial t} \end{aligned}\right\} —$$

are called Gauge transformation and arbitrary scalar Λ is Gauge function

we say that field vector are invariant to Gauge transformation i.e. they are Gauge invariant.

Lorentz Gauge

The vector $\vec{A}$ is not completely specified by giving only its curl but if both the curl and divergence of a vector are specified, the vector is uniquely determined. Generally div A is chosen in two ways in accordance with the field containing the charge or not.

Maxwell's field eq^ns in terms of e.m. potentials

$$\nabla^2 \vec{A} - \mu\epsilon \frac{\partial^2 \vec{A}}{\partial t^2} - \nabla\left(\nabla\cdot\vec{A} + \mu\epsilon\frac{\partial\phi}{\partial t}\right) = -\mu\vec{J} \quad \text{—(1)}$$

$$\nabla^2 \phi - \mu\epsilon\frac{\partial^2\phi}{\partial t^2} + \frac{\partial}{\partial t}\left(\nabla\cdot\vec{A} + \mu\epsilon\frac{\partial\phi}{\partial t}\right) = -\rho/\epsilon_0 \quad \text{—(2)}$$

Let us choose

$$\nabla\cdot\vec{A} + \mu\epsilon\frac{\partial\phi}{\partial t} = 0 \quad \text{—(3)}$$

then eq^n (1) and (2) will become identical and uncoupled

~~therefore~~ This requirement is called Lorentz condition

If the vector and scalar potential satisfy it the Gauge is known as Lorentz Gauge.

Applying Lorentz condition in eq^n (1) and (2) we get

$$\nabla^2 \vec{A} - \mu\epsilon \frac{\partial^2 \vec{A}}{\partial t^2} = -\mu \vec{J} \quad \text{— (4)}$$

and
$$\nabla^2 \phi - \mu\epsilon \frac{\partial^2 \phi}{\partial t^2} = -\rho/\epsilon \quad \text{— (5)}$$

above eq^n's become

$$\Box^2 \vec{A} = -\mu \vec{J} \quad \text{— (6)}$$

$$\Box^2 \phi = -\rho/\epsilon \quad \text{— (7)}$$

where $\Box^2 = \nabla^2 - \frac{1}{v^2}\frac{\partial^2}{\partial t^2}$ $\left\{ \because v^2 = \frac{1}{\mu\epsilon} \right.$

Eq^n (6) and (7) are known as D'Alembertian eq^n's and can be solved in general as solved as poisson eq^n.

The potential obtained by solving these eq's are called retarded potential.

The Lorentz condition places on Λ, A' and ϕ' are replaced as

$$A' = A + \nabla \Lambda$$

$$\phi' = \phi - \frac{\partial \Lambda}{\partial t}$$

using eq^n (3), we get

$$\nabla \cdot (A' - \nabla \Lambda) + \mu\epsilon \frac{\partial}{\partial t}\left(\phi' - \frac{\partial \Lambda}{\partial t}\right) = 0$$

$$\nabla \cdot A' + \mu\epsilon \frac{\partial \phi'}{\partial t} = \nabla^2 \Lambda - \mu\epsilon \frac{\partial^2 \Lambda}{\partial t^2}$$

so A' and ϕ' will also satisfy eq^n (3) i.e. Lorentz condition if

Replacing $\rho(r',t)$ by $\vec{J}(r',t)$, we get

$$\nabla^2\left\{\frac{1}{4\pi}\int\frac{J(r',t)}{R}d\tau'\right\} = -\frac{\vec{J}(r',t)}{c} \quad — (8)$$

Taking $\vec{G} = \int\frac{\vec{J}}{R}d\tau'$, and using vector identity

$$\nabla\times(\nabla\times\vec{G}) = \nabla(\nabla\cdot\vec{G}) - \nabla^2\vec{G}$$

$$\nabla^2\vec{G} = \nabla(\nabla\cdot\vec{G}) - \nabla\times(\nabla\times G)$$

we get

$$\nabla^2\int\frac{J(r',t)}{R}d\tau' = \nabla\left\{\nabla\cdot\int\frac{\vec{J}(r',t)}{R}d\tau'\right\}$$
$$-\nabla\times\left\{\nabla\wedge\int\frac{\vec{J}(r',t)}{R}d\tau'\right\}$$

Eqn (8) reduced to

$$-4\pi\vec{J}(r',t) = \nabla\left\{\nabla\cdot\int\frac{\vec{J}(r',t)}{R}d\tau'\right\} - \nabla\times\left\{\nabla\times\int\frac{J(r',t)}{R}d\tau'\right\}$$

or $$\vec{J}(r',t) = -\frac{1}{4\pi}\nabla\left\{\nabla\cdot\int\frac{\vec{J}(r',t)}{R}d\tau'\right\} + \frac{1}{4\pi}\nabla\times\nabla\times\int\frac{J(r',t)}{R}d\tau' \quad — (9)$$

we can write

$$\nabla\cdot\int\frac{\vec{J}(r',t)}{R}d\tau = \int\vec{J}(r',t)\cdot\nabla\left(\frac{1}{R}\right)d\tau'$$

$$= -\int J(r',t)\cdot\nabla'\left(\frac{1}{R}\right)d\tau' \quad \left\{\begin{array}{l}\nabla\left(\frac{1}{R}\right) = -\nabla'\left(\frac{1}{R}\right)\\ R^2 = (x-x')^2+(y-y')^2+(z-z')^2\end{array}\right.$$

$$= -\int\left[\nabla'\cdot\left(\frac{\vec{J}(r',t)}{R}\right) - \frac{\nabla'\cdot J(r',t)}{R}\right]d\tau'$$

[since $\operatorname{div}(\vec{A}\vec{B}) = \vec{A}\nabla\cdot\vec{B} + \vec{B}\cdot\nabla\vec{A}$]

$$= -\int \frac{\nabla' J(r',t)}{R} d\tau' - \int \nabla' \cdot \left(\frac{\vec{J}(r',t)}{R} \right) d\tau'$$

$$= \int \frac{\nabla' \cdot J(r',t)}{R} d\tau' - \oint \frac{\vec{J}(r',t)}{R} ds$$

$$= \int \frac{\nabla' \cdot \vec{J}(r',t)}{R} d\tau' \quad — (10)$$

as in limit $R \to \infty$, second term will vanish

Eq^n (9) can be written as

$$\vec{J}(r',t) = -\frac{1}{4\pi} \nabla \int \frac{\nabla' \cdot J(r',t)}{R} d\tau' + \frac{1}{4\pi} \nabla \times \nabla \times \int \frac{\vec{J}(r',t)}{R} d\tau'$$

i.e. $\vec{J} = \vec{J}_l + \vec{J}_t \quad — (11)$

where

$$\vec{J}_l = -\frac{1}{4\pi} \nabla \int \frac{\nabla' \cdot \vec{J}(r',t)}{R} d\tau' \quad — (12)$$

$$J_t = \frac{1}{4\pi} \nabla \times \nabla \times \int \frac{\vec{J}(r',t)}{R} d\tau' \quad — (13)$$

Now

$$\nabla \times \vec{J}_l = \nabla \times \left\{ -\frac{1}{4\pi} \nabla \int \frac{\nabla' \cdot J(r',t)}{R} d\tau' \right\}$$

$$= -\frac{1}{4\pi} \text{curl grad} \left\{ \int \frac{\nabla' \cdot J}{R} d\tau' \right\} = 0 \quad — (14)$$

$$\nabla \cdot \vec{J}_t = \nabla \cdot \left\{ \nabla \times \nabla \times \frac{1}{4\pi} \int \frac{\vec{J}(r',t)}{R} d\tau' \right\}$$

$$= \frac{1}{4\pi} \text{div curl} \left\{ \nabla \times \int \frac{\vec{J}(r',t)}{R} \right\} = 0 \quad — (15)$$

$$\nabla^2 \Lambda - \mu\epsilon \frac{\partial^2 \Lambda}{\partial t^2} = 0$$

$$\Box^2 \Lambda = 0 \quad \text{—— (8)}$$

i.e. Lorentz condition is invariant under those Gauge transformation for which Gauge functions are solutions of that homogeneous wave eq^{n}s.

Advantages of Lorentz Gauge

(i) It makes the eq^{n}s for $\vec{A}$ and ϕ independent of each other

(ii) It leads to the wave eq^{n}s which treat ϕ and $\vec{A}$ on equivalent footings.

(iii) It is a concept which is independent of the coordinate system chosen and so fits naturally into the considerations of special theory of relativity.

Coulomb Gauge

From field eq^n in e.m. potential

$$\nabla^2\vec{A} - \mu\epsilon\frac{\partial^2\vec{A}}{\partial t^2} - \nabla\left(\nabla\cdot\vec{A} + \mu\epsilon\frac{\partial\phi}{\partial t}\right) = -\mu J \quad \text{— (1)}$$

and
$$\nabla^2\phi - \mu\epsilon\frac{\partial^2\phi}{\partial t^2} + \frac{\partial}{\partial t}\left(\nabla\cdot\vec{A} + \mu\epsilon\frac{\partial\phi}{\partial t}\right) = -\rho/\epsilon \quad \text{— (1)}$$

or
$$\nabla^2\phi + \frac{\partial}{\partial t}(\nabla\cdot\vec{A}) = -\rho/\epsilon \quad \text{—— (2)}$$

Show that if we assume

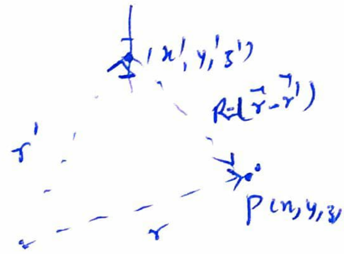
$$\operatorname{div} A = \nabla \cdot A = 0 \quad — (3)$$

the eqn (2) reduces to poisson eqs

$$\nabla^2 \phi = -\rho/\epsilon$$

or

$$\nabla^2 \phi(x, y, z) = -\frac{\rho(x', y', z', t)}{\epsilon} \quad — (4)$$



solution of it is

$$\phi(x, y, z) = \frac{1}{4\pi\epsilon} \int \frac{\rho(x', y', z', t)}{R} \, d\tau' \quad — (5)$$

i.e. scalar potential is as Coulomb's potential due to charge density $\rho(x', y', z', t)$

this is the origin of the name Coulomb Gauge.

Eqn (1) becomes using Coulomb's Gauge

$$\nabla^2 A - \frac{1}{v^2} \frac{\partial^2 \vec{A}}{\partial t^2} = -\mu \vec{J} + \mu\epsilon \nabla\left(\frac{\partial \phi}{\partial t}\right) \quad — (6)$$

using eqn (5), eqn (4) becomes

$$\nabla^2 \left\{ \frac{1}{4\pi\epsilon} \int \frac{\rho(r', t)}{R} \, d\tau' \right\} = -\frac{\rho(r', t)}{\epsilon} \quad — (7)$$

the poisson's eqn hold good for scalar and vector both

The first term on R.H.S of eqn (11) is irrotational and second is solenoidal.

The first term is called longitudinal current and second is transverse current.

Using (11), eqn (6) can be written as

$$\nabla^2\vec{A} - \frac{1}{v^2}\frac{\partial^2\vec{A}}{\partial t^2} = -\mu(\vec{J}_l + \vec{J}_t) + \mu\epsilon\nabla\left(\frac{\partial\phi}{\partial t}\right)$$

$$= -\mu\vec{J}_l - \mu\vec{J}_t + \mu\epsilon\nabla\left\{\frac{\partial}{\partial t}\left(\frac{1}{4\pi\epsilon}\int\frac{\rho}{R}\,d\tau'\right)\right\}$$

$$= -\mu\vec{J}_l - \mu\vec{J}_t + \frac{\mu}{4\pi}\nabla\int -\frac{\nabla'\cdot\vec{J}}{R}\,d\tau'$$

$$\left\{ \frac{\partial\rho(r',t)}{\partial t} = -\nabla'\cdot\vec{J} \quad \text{continuity eqn} \right.$$

$$= -\mu\vec{J}_l - \mu\vec{J}_t + \mu\vec{J}_l$$

$$= -\mu\vec{J}_t$$

or $\quad \Box^2\vec{A} = -\mu\vec{J}_t \quad$ —— (16)

Thus the eqn for $\vec{A}$ can be expressed entirely in terms of the transverse current.

So this Gauge is also called transverse Gauge.

Advantage of Coulomb Gauge

(i) In Coulomb Gauge, the scalar potential is exactly the electrostatic potential { eqn (5) }

(ii) The electric field $\vec{E}$ as per eqn

$$\vec{E} = -\nabla\phi - \frac{\partial \vec{A}}{\partial t}$$

is separated into electrostatic field $V = \phi$ and a wave field given by $-\frac{\partial \vec{A}}{\partial t}$

(iii) The Gauge is often used when no source are present. Then following eqn (11) $\phi = 0$ and $\vec{A}$ satisfies the homogeneous wave eqn (16)

The fields are given by $\vec{E} = -\frac{\partial \vec{A}}{\partial t}$

and $\vec{B} = \nabla \times \vec{A}$

Plane-wave solution for Maxwell's equations

Derivation of e.m. equation by use of Maxwell's equation is one of the important application of them.

Consider the uniform linear medium

permitivity → ϵ

permeability → μ ~~and~~

conductivity → σ

then Displacement current

$$\vec{D} = \epsilon\vec{E}, \vec{B} = \mu H, J = \sigma E \text{ and } \rho = 0 \quad \text{—(1)}$$

There is no charge other than as derived by

Ohm's law,

The Maxwell's eqn's are

$$\left.\begin{array}{ll} \nabla\cdot\vec{D} = \rho & \nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \\ \nabla\cdot\vec{B} = 0 & \nabla\times\vec{H} = \vec{J} + \frac{\partial\vec{D}}{\partial t} \end{array}\right\} \quad \text{—(2)}$$

then current case ($\rho = 0$, and $D = \epsilon E$, $B = \mu H$, $J = \sigma E$)

$$\nabla\cdot\vec{E} = 0 \quad \text{— (3)}$$

$$\nabla\cdot\vec{H} = 0 \quad \text{— (4)}$$

$$\nabla\times\vec{E} = -\mu\frac{\partial H}{\partial t} \quad \text{— (5)}$$

$$\nabla\times\vec{H} = \sigma\vec{E} + \epsilon\frac{\partial\vec{E}}{\partial t} \quad \text{— (6)}$$

taking curl of eqn (5), we get

$$\nabla\times(\nabla\times\vec{E}) = -\mu\frac{\partial}{\partial t}(\nabla\times\vec{H})$$

then

$$\nabla\times(\nabla\times\vec{E}) = -\mu\frac{\partial}{\partial t}\left(\sigma\vec{E}+\epsilon\frac{\partial\vec{E}}{\partial t}\right)$$

$$\nabla\times(\nabla\times\vec{E}) = -\sigma\mu\frac{\partial\vec{E}}{\partial t} - \mu\epsilon\frac{\partial^2\vec{E}}{\partial t^2} \quad \text{--- (7)}$$

Similarly taking the curl of eqn (6), we get

$$\nabla\times(\nabla\times H) = -\sigma\mu\frac{\partial\vec{H}}{\partial t} - \mu\epsilon\frac{\partial^2\vec{H}}{\partial t^2} \quad \text{--- (8)}$$

Using vector identity

$$\nabla\times(\nabla\times\vec{A}) = \nabla(\nabla\cdot\vec{A}) - \nabla^2\vec{A}$$

from eqn (7) and (8), we get

$$\nabla^2\vec{E} = \sigma\mu\frac{\partial\vec{E}}{\partial t} + \mu\epsilon\frac{\partial^2\vec{E}}{\partial t^2} \quad \text{--- (9)}$$

and

$$\nabla^2 H = \sigma\mu\frac{\partial\vec{H}}{\partial t} + \mu\epsilon\frac{\partial^2\vec{H}}{\partial t^2} \quad \text{--- (10)}$$

Since $\nabla(\nabla\cdot E) =$ $\nabla(\nabla\cdot H) =$

Eqn's (9) and (10) are wave eqn, which govern the e.m. field in a homogeneous linear medium in which $\rho=0$

Any field which ~~satisfies~~ solution of maxwell's eqn satisfies the eqn (9) and (10), But ~~so~~ it is not necessary that solutions of eqn (9) and (10) satisfy the Maxwell's eqn's.

we obtain the solutions of eqn (9) and (10) in such manner that Maxwell's eqn are also satisfied.

~~we know that~~

This problem can be solved using the method of complex variable. ~~for~~

Let time dependence of field vector is

$$\vec{E}(r,t) = \vec{E}_s(r) e^{i\omega t} \quad \text{— (11)}$$

Using eq (9) and (10)

$$\nabla^2 E_s = \omega^2 \mu \epsilon E_s + i\omega \sigma \mu E_s = 0 \quad \text{— (12)}$$

Since E_s is real part of $\vec{E}$

The spatial Electric field depend on the space coordinates

$$\vec{E}_s = \vec{E}_s(r) \quad \text{— (13)}$$

for plane e.m. waves is

$$\vec{E}_s = \vec{E}_0 e^{j\vec{k}\cdot\vec{r}} \quad \text{— (14)}$$

where $\vec{k}$ is propagation vector

$$\vec{k} = \left(\frac{2\pi}{\lambda}\right)\hat{n} = \frac{\omega}{\vec{v}}\hat{n} \quad \text{— (15)}$$

where $\hat{n}$ is unit vector along $\vec{k}$

and $\vec{v} \Rightarrow$ phase velocity

then eq' (11) is

$$\vec{E}(\vec{r},t) = E_0 e^{i\vec{k}\cdot\vec{r} - i\omega t} \quad \text{— (16)}$$

here $\vec{E}_0$ is complex amplitude and constant in space and time when field vector is in form of eqn (16) then

$$\left.\begin{aligned} \text{grad} &\rightarrow \cancel{\nabla} \rightarrow i\vec{k} \\ \text{div} &\rightarrow \nabla\cdot \rightarrow ik\cdot \\ \text{Curl} &\rightarrow \nabla\times \rightarrow ik\times \\ \text{and } \frac{\partial}{\partial t} &\rightarrow -i\omega \end{aligned}\right\} \quad \text{— (17)}$$

(1) E.M. wave in free space

for free space

$\rho = 0$, $\sigma = 0$, $\mu = \mu_0$ and $\epsilon = \epsilon_0$

then eqn (9) and (10) becomes

$$\left.\begin{aligned} \nabla^2\vec{E} = \mu_0\epsilon_0 \frac{\partial^2\vec{E}}{\partial t^2} &\Rightarrow \nabla^2\vec{E} - \mu_0\epsilon_0 \frac{\partial^2\vec{E}}{\partial t^2} = 0 \\ \text{and } \nabla^2\vec{H} = \mu_0\epsilon_0 \frac{\partial^2\vec{H}}{\partial t^2} &\Rightarrow \nabla^2\vec{H} - \mu_0\epsilon_0 \frac{\partial^2\vec{H}}{\partial t} = 0 \end{aligned}\right\} \quad (18)$$

Eqn (18) represent's wave eqn's governing E.M. field in free space.

$$\mu_0\epsilon_0 = \frac{1}{v^2} = \frac{1}{c^2}$$

where $c = 3\times10^8$ m/s so the field vectors $\vec{E}$ and $\vec{H}$ are propagated in free space as wave at a speed equal to speed of light.

plane wave → wave whose amplitude is same at any point in a perpendicular to a specified direction. the $\vec{E}$ and $\vec{H}$ field are both perpendicular to the direction of $\vec{k}$. the E.M. wave is transverse in nature.

(iii) E. M. waves in non-conducting isotropic medium

for non-conducting media

$$\vec{J} = \sigma \vec{E} = 0$$
$$\rho = 0$$
$$\vec{D} = \epsilon \vec{E} \text{ and } \vec{B} = \mu \vec{H}$$

from eqn (9) and (10)

$$\nabla^2 \vec{E} = \sigma \mu \frac{\partial \vec{E}}{\partial t} + \mu\epsilon \frac{\partial^2 \vec{E}}{\partial t^2} \quad \text{— (9)}$$

$$\nabla^2 \vec{H} = \sigma \mu \frac{\partial \vec{H}}{\partial t} + \mu\epsilon \frac{\partial^2 \vec{H}}{\partial t^2} \quad \text{— (10)}$$

Using above value, we get

$$\nabla^2 \vec{E} = \mu\epsilon \frac{\partial^2 \vec{E}}{\partial t^2} \Rightarrow \nabla^2 \vec{E} - \mu\epsilon \frac{\partial^2 \vec{E}}{\partial t^2} \quad \text{— (19)}$$

$$\nabla^2 \vec{H} = \mu\epsilon \frac{\partial^2 \vec{H}}{\partial t^2} \Rightarrow \nabla^2 \vec{H} - \mu\epsilon \frac{\partial^2 \vec{H}}{\partial t^2} \quad \text{— (20)}$$

eqn (19) and (20) are vector eqn of identical form. therefore each of the six components of $\vec{E}$ and $\vec{H}$ separately satisfies the scalar wave eqn of the form.

$$\nabla^2 u - \mu\epsilon \frac{\partial^2 u}{\partial t^2} = 0 \quad \text{— (21)}$$

Where u is scalar and can stand for any one components of $\vec{E}$ and $\vec{H}$

eqn (21) is resemble as general wave eqn

$$\nabla^2 u - \frac{1}{v^2} \frac{\partial^2 u}{\partial t^2} = 0 \quad \text{— (22)}$$

Where v is speed of wave.

$$v = \frac{1}{\sqrt{\mu\epsilon}} \quad \text{--- (23)}$$

It indicates that field vector $\vec{E}$ and $\vec{H}$ are propagated in non-conducting medium as wave with speed v

$$v = \frac{1}{\sqrt{\mu\epsilon}} = \frac{1}{\sqrt{\epsilon_r\epsilon_0 \mu_m \mu}} = \frac{1}{\sqrt{k_e\epsilon_0 K_m \mu_0}} \quad \text{--- (24)}$$

where

$k_m = \mu_m \rightarrow$ relative permeability of medium

and $k_e = \epsilon_r$ is relative permittivity of medium (dielectric constant)

as $\frac{1}{\sqrt{\mu_0\epsilon_0}} = c$ speed of e.m. wave in free space

$$v = \frac{c}{\sqrt{K_e K_m}} \quad \text{--- (25)}$$

Since $k_e > 1$ and $k_m > 1$

then speed of wave in medium is less than speed of wave in free space i.e.

$$v < c$$

in term of refractive index (η)

we know that

$$\eta = \frac{c}{v} \Rightarrow v = \frac{c}{\eta} \quad \text{--- (26)}$$

from (25) and (26)

$$\eta = \sqrt{k_e k_m} \quad \text{--- (27)}$$

For non-magnetic medium $k_m = 1$ then

$$\left.\begin{aligned}\eta &= \sqrt{k_e} \\ \eta^2 &= k_e\end{aligned}\right\} \quad \text{--- (28)}$$

This eq^n (28) is known as Maxwell's relation

Again from eq^n (19) and (20)

$$\nabla^2\vec{E} - \frac{1}{v^2}\frac{\partial^2\vec{E}}{\partial t^2} = 0 \quad \text{--- (29)}$$

$$\nabla^2\vec{H} - \frac{1}{v^2}\frac{\partial^2\vec{H}}{\partial t^2} = 0 \quad \text{--- (30)}$$

The ~~p~~ plane wave solutions of eq^n (29) and (30) ~~acts~~ are

$$\vec{E}(\vec{r},t) = \vec{E_0}\, e^{i(\vec{K}\cdot\vec{r} - \omega t)} \quad \text{--- (31)}$$

$$\vec{H}(\vec{r},t) = \vec{H_0}\, e^{i(\vec{K}\cdot\vec{r} - \omega t)} \quad \text{--- (32)}$$

k is wave propagation vector is

$$\vec{k} = k\hat{n} = \frac{\omega}{v}\hat{n} \quad \text{--- (33)}$$

Relative direction of $\vec{E}$ and $\vec{H}$ (orthogonality of $\vec{E}$, $\vec{H}$ and $\vec{k}$)

The requirement $\nabla\cdot\vec{E} = 0$

$\nabla\cdot\vec{H} = 0$

$$\begin{cases}\nabla\cdot\vec{E_0}\, e^{i(\vec{k}\cdot\vec{r} - \omega t)} = 0 \\ \nabla\cdot\vec{H_0}\, e^{i(\vec{k}\cdot\vec{r} - \omega t)} = 0\end{cases}$$

~~dep~~ demand that

$$\left.\begin{aligned}\vec{k}\cdot\vec{E} &= 0 \\ \vec{k}\cdot\vec{H} &= 0\end{aligned}\right\} \quad \text{--- (34)}$$

$$\begin{cases}\nabla\cdot \rightarrow i\vec{k}\cdot \\ \nabla\times \rightarrow i\vec{k}\times\end{cases}$$

Therefore the directions of both $\vec{E}$ and $\vec{H}$ are perpendicular of $\vec{K}$

The e.m. waves in dielectric medium are transverse in nature.

The curl of $\vec{E}$ and $\vec{H}$ (maxwell's IIIrd and IV eq^n)

$$\left.\begin{aligned}\nabla\times\vec{E} &= -\mu\frac{\partial \vec{H}}{\partial t}\\ \nabla\times\vec{H} &= \epsilon\frac{\partial E}{\partial t}\end{aligned}\right\} \quad -(35)$$

using eq^n (31) and (32)

$$\nabla\times\vec{E}_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)} = -\mu\frac{\partial}{\partial t}\vec{H}_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$

we get

$$\vec{k}\times\vec{E} = \mu\omega\vec{H} \quad -(36)$$

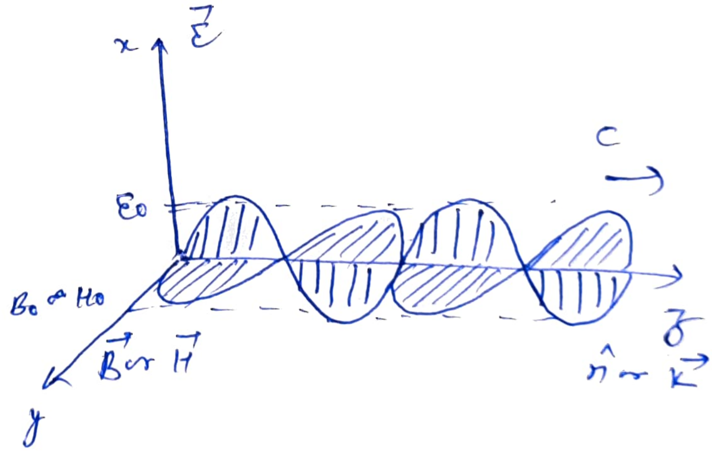
Similarly

$$\nabla\times H_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)} = +\epsilon\frac{\partial}{\partial t}E_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$

we get $\vec{k}\times\vec{H} = -\epsilon\omega\vec{E} \quad -(37)$

Eq^n (36) and (37) indicate that field vector $\vec{E}$ and $\vec{H}$ are mutually perpendicular to the $\vec{k}$

The vector $\vec{E}$, $\vec{H}$ and $\vec{k}$ form a set of orthogonal vector as shown in figure



<u>phase of $\vec{E}$ and $\vec{H}$ and</u>

<u>wave impedance</u>

from eq^n (36)

$$H = \frac{1}{\mu\omega}(\vec{k}\times\vec{E})$$

$$= \frac{k}{\mu\omega}(\hat{n}\times\vec{E})$$

$$= \frac{\omega}{v\mu\omega}(\hat{n}\times\vec{E})$$

since $\vec{k} = k\hat{n}$

$k = \frac{\omega}{v}$

$$\vec{H} = \frac{1}{\mu v}(\hat{n}\times\vec{E})$$

$$= \sqrt{\frac{\epsilon}{\mu}}\,(\hat{n}\times\vec{E}) \quad — (38) \qquad \left\{ v = \frac{1}{\sqrt{\mu\epsilon}} = \right.$$

Similarly

$$\vec{E} = -\frac{1}{\epsilon\omega}(\vec{k}\times\vec{H})$$

$$= -\frac{1}{\epsilon\omega}k\,(\hat{n}\times\vec{H})$$

$$= -\frac{1}{\epsilon v}(\hat{n}\times\vec{H})$$

$$= -\sqrt{\frac{\mu}{\epsilon}}\,(\hat{n}\times\vec{H}) \quad — (39)$$

The ratio of magnitude of $\vec{E}$ and magnitude of $\vec{H}$ is defined as impedance of wave and denoted by Z

$$Z = \frac{|\vec{E}|}{|\vec{H}|} = \frac{|E_0|}{|H_0|} \quad — (40)$$

Let wave is propagating in z direction then from eq^n (36)

$$\vec{k}\times\vec{E} = \mu\omega\vec{H} \qquad \left\{ \vec{k} = k_z\hat{z} \right.$$

$$k_z\hat{z}\times(E_x\hat{x} + \ldots) = \mu\omega H_y\hat{y}$$

$$k_z E_x\hat{y} = \mu\omega H_y\hat{y}$$

$$\frac{E_x}{H_y} = \frac{\mu\omega}{k_z} = \frac{\mu\omega}{\omega/v_x} = \mu v_x \qquad \left\{ \begin{array}{l} E_x = E_{0x}e^{i(k_z z - \omega t)} \\ H_y = H_{0y}e^{i(k_z z - \omega t)} \end{array} \right.$$

$$\frac{E_x}{H_y} = \frac{E_{0x}}{H_{0y}} = \sqrt{\frac{\mu}{\epsilon}} \quad — (41)$$

in general

$$Z = \frac{E_0}{H_0} = \sqrt{\frac{\mu}{\epsilon}}$$

$$Z = \sqrt{\frac{K_m \mu_0}{K_e \epsilon_0}} = \text{real quantity} \quad \text{--- (42)}$$

the field vector $\vec{E}$ and $\vec{H}$ are in same phase, they have the same relative magnitude at all point at all time.

the unit of 'Z' comes out to be ohm

$$Z = \frac{\vec{E}}{\vec{H}} = \frac{V/m}{A/m} = \frac{V}{A} = \text{ohm}$$

hence Z is called as wave impedance of dielectric medium

The ~~impede~~ impedance of free space

$$Z = \sqrt{\frac{K_m \mu_0}{K_e \epsilon_0}} = Z_0 \sqrt{\frac{K_m}{K_e}} \quad \text{--- (43)}$$

$$Z_0 = \sqrt{\frac{\mu_0}{\epsilon_0}} = 377 \text{ ohm}$$

$Z_0 \rightarrow$ impedance of free space

Poynting theorem

Electrostatic energy is

$$U_e = \frac{1}{2}\int_v (\vec{E}\cdot\vec{D})\, d\tau \quad — (1)$$

Electromagnetic energy is (energy stored in M field)

$$U_M = \frac{1}{2}\int_v (\vec{H}\cdot\vec{B})\, d\tau \quad — (2)$$

From Maxwell eqn in vacuum

$$\left.\begin{aligned} \nabla\cdot\vec{D} &= 0 \\ \nabla\cdot\vec{B} &= 0 \\ \nabla\times\vec{E} &= -\frac{\partial\vec{B}}{\partial t} \\ \nabla\times\vec{H} &= \vec{J} + \frac{\partial\vec{D}}{\partial t} \end{aligned}\right\} \quad — (3)$$

Taking scalar product of ~~with~~ $\vec{III}$ of eqn (3) with $\vec{H}$ and ~~with~~ IV of eqn (3) with $\vec{E}$

$$\left.\begin{aligned} \vec{H}\cdot(\nabla\times\vec{E}) &= -\vec{H}\cdot\frac{\partial\vec{B}}{\partial t} \\ \text{and} \quad \vec{E}\cdot(\nabla\times\vec{H}) &= \vec{E}\cdot\vec{J} + \vec{E}\cdot\frac{\partial\vec{D}}{\partial t} \end{aligned}\right\} \quad — (4)$$

we get

$$\vec{H}\cdot(\nabla\times\vec{E}) - \vec{E}\cdot(\nabla\times\vec{H}) = -\vec{H}\cdot\frac{\partial\vec{B}}{\partial t} - \vec{E}\cdot\frac{\partial\vec{D}}{\partial t} - \vec{E}\cdot\vec{J}$$

$$= -\left[\vec{H}\cdot\frac{\partial\vec{B}}{\partial t} + \vec{E}\cdot\frac{\partial\vec{D}}{\partial t}\right] - \vec{E}\cdot\vec{J} \quad — (5)$$

Using vector identity

$$\nabla\cdot(\vec{E}\times\vec{H}) = \vec{H}\cdot(\nabla\times\vec{E}) - \vec{E}\cdot(\nabla\times\vec{H})$$

we get

$$\nabla\cdot(\vec{E}\times\vec{H}) = -\left[\vec{H}\cdot\frac{\partial\vec{B}}{\partial t} + \vec{E}\cdot\frac{\partial\vec{D}}{\partial t}\right] - \vec{E}\cdot\vec{J} \quad — (6)$$

using $\vec{B} = \mu \vec{H}$ and $D = \epsilon \vec{E}$

then from above eqn

$$\vec{E} \cdot \frac{\partial \vec{D}}{\partial t} = \epsilon \frac{\partial}{\partial t}(E\epsilon) = \frac{1}{2}\epsilon \frac{\partial}{\partial t}(\vec{E}^2) = \frac{1}{2}\frac{\partial}{\partial t}(\vec{E}\cdot\epsilon\vec{E})$$

$$= \frac{\partial}{\partial t}\left(\frac{1}{2}\vec{E}\cdot\vec{D}\right)$$

and $$\vec{H} \cdot \frac{\partial \vec{B}}{\partial t} = \vec{H}\frac{\partial}{\partial t}(\mu\vec{H}) = \frac{\partial}{\partial t}\left(\frac{1}{2}\vec{H}\cdot\vec{B}\right)$$

eqn (6) becomes

$$\nabla\cdot(\vec{E}\times\vec{H}) = -\frac{\partial}{\partial t}\left(\frac{1}{2}(\vec{E}\cdot\vec{D} + \vec{H}\cdot\vec{B})\right) - \vec{J}\cdot\vec{E} \quad — (7)$$

on integrating over volume V bounded by surface s

$$\oint_V \nabla\cdot(\vec{E}\times\vec{H})\,d\tau = -\int_V \frac{\partial}{\partial t}\left[\frac{1}{2}\{(\vec{E}\cdot\vec{D}) + (\vec{H}\cdot\vec{B})\}\right]d\tau - \int_V \vec{J}\cdot\vec{E}\,d\tau$$

$$\oint_s (\vec{E}\times\vec{H})\,ds = -\frac{d}{dt}\int_V \frac{1}{2}(\vec{E}\cdot\vec{D} + \vec{H}\cdot\vec{B})\,d\tau - \int_V \vec{J}\cdot\vec{E}\,d\tau$$

$$-\int_V \vec{J}\cdot\vec{E}\,d\tau = \frac{d}{dt}\int_V \frac{1}{2}(\vec{E}\cdot\vec{D} + \vec{H}\cdot\vec{B})\,d\tau + \oint_s(\vec{E}\times\vec{H})\,d\tau \quad — (8)$$

Consider a charge q moving with velocity v under the ~~comp~~ combined effect of mechanical, electrical and magnetic force, i.e. lorentz force on charge 'q' $\Rightarrow F = q(E + \vec{v}\times\vec{B})$

workdone by force

$$\delta = \vec{F}\cdot\vec{dl} = q(\vec{E} + \vec{v}\times\vec{B})\cdot\vec{v}\,dt$$

$$dW = q\,\vec{E}\cdot\vec{v}\,dt$$

$$\frac{dW}{dt} = q\vec{E}\cdot\vec{v} = \vec{E}\cdot q\vec{v}$$

We know that $q = \int \rho \, d\tau$

ρ is charge density

then

$$\frac{dW}{dt} = \int_V \vec{E}\cdot\vec{v}\rho \, d\tau \quad -(9)$$

and $\rho v = \vec{J}$

Eq^n (9) becomes

$$\frac{dW}{dt} = \int_V \vec{E}\cdot\vec{J} \, d\tau \quad -(10)$$

It represents power density that is transferred into E.M. field. then $-\int \vec{E}\cdot\vec{J} \, d\tau$ represents rate of energy transferred into e.m. field through the motion of free charge in V

~~Using~~ eq^n (8) becomes, ~~we get~~ using eq^n (10) and (1) and (2)

$$\frac{dW}{dt} = -\frac{d}{dt}\int_V \frac{1}{2}\left(\epsilon_0 E^2 + \frac{1}{\mu}B^2\right) d\tau - \frac{1}{\mu}\oint_S (\vec{E}\times\vec{B}) \, ds$$

$$= \int_V \frac{1}{2}(\vec{E}\cdot\vec{D} + \vec{B}\cdot\vec{B}) \, d\tau - \oint_S (\vec{E}\times\vec{H}) \, ds \quad -(11)$$

or

$$\frac{dW}{dt} = -\frac{d}{dt}\int_V \frac{1}{2}(\vec{E}\cdot\vec{D}$$

This is called ~~Pooty~~ Poynting theorem. and also called work-energy theorem of "electrodynamics"

The first integral on R.H.S. is total energy stored in field.

The second term represents the rate ~~of~~ at which energy is carried out of V, across the boundary surface by e.m. field.

The Poynting theorem says" → The work done on the charges by e.m. force is equal to the decrease in energy stored in field less the energy that flowed out through the surface.

The energy per unit time per unit area transported by field is called Poynting vector

$$\vec{S} = \frac{1}{\mu}(\vec{E}\times\vec{B}) = \vec{E}\times\vec{H} \quad \text{—} (12)$$

$\vec{S}$ is energy flux density

from eq^n (11)

$$\frac{dW}{dt} = -\frac{d}{dt}(U_e + U_m) - \oint_s \vec{S}\cdot d\vec{s} \quad \text{—}$$

$$\frac{dW}{dt} = -\frac{d}{dt}(U_{em}) - \oint_s \vec{S}\cdot ds \quad \text{—} (13)$$

The work done on charge will increase their mechanical energy (K.E & P.E.).

then $$\frac{dW}{dt} = \frac{d}{dt}\int_v U_{mech}\, d\tau \quad \text{—} (14)$$

then $$\frac{d}{dt}\int_v (U_{mech} + U_{em})\, d\tau = -\oint_s \vec{S}\cdot d\vec{s} = -\int_v (\nabla\cdot\vec{S})\, d\tau$$

$$\frac{\partial}{\partial t}(U_{mech} + U_{em}) = -\nabla\cdot\vec{S} \quad \text{—} (15)$$

This is differential form of Poynting theorem.

Compared by continuity eq^n

$$\frac{\partial \rho}{\partial t} = -\nabla\cdot\vec{J}$$

Then charged density is replaced by energy density (mechanical & e.m) and current density by Poynting vector.

The flow of energy is represents same as flow of charge.