UNIT V

Mathematical expectation :-

Let X be a random variable taking n values as $x_1, x_2 - - x_n$ with respective probablities as $p_1, p_2 - - p_n$ such that $\sum_{i=1}^{n} p_i = 1$ then Mathematical expectation or simply expectation of r.v. X is defined as

$$E(X) = x_1 p_1 + x_2 p_2 + x_3 p_3 + - - - + x_n p_n$$

or $\boxed{E(X) = \sum_{i=1}^{n} x_i p_i \text{, (for discrete r.v.)}}$

or $E(X) = \Sigma x p$

This E(X) is also called mean or arithmatic mean of r.v. X & it is also denoted by $\overline{X}$

$$E(X) = \overline{X}$$

(i.e. if we have to find mean = we find expectation)

Ex.

	x_1	x_2	x_3	x_4	x_5	x_6
X :	1	2	3	4	5	6
p :	1/6	1/6	1/6	1/6	1/6	1/6

then $E(X) = 1\times\frac{1}{6} + 2\times\frac{1}{6} + 3\times\frac{1}{6} + 4\times\frac{1}{6} + 5\times\frac{1}{6} + 6\times\frac{1}{6}$

$= \frac{1}{6} + \frac{2}{6} + \frac{3}{6} + \frac{4}{6} + \frac{5}{6} + \frac{6}{6}$

$= \frac{21}{6} = 3.5$

The Mathematical expectation of a continous random variable x with probablity density functions $f(x)$, is as follows

$$\boxed{E(X) = \int_{-\infty}^{\infty} x\, f(x)\, dx, \text{ (for cont. case)}}$$

Expected value of a function of a Random Variable :-

Let $\phi(X)$ be a function of r.v. X then $E[\phi(X)]$ is defined as.

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$$E[\phi(X)] = \sum_{x} \phi(x_i)\, p(x) \text{, (for discrete)}$$

$$E[\phi(x)] = \int_{-\infty}^{\infty} \phi(x_i)\, f(x)\, dx \text{, (for cont.)}$$

If we take $\phi(X) = X^r$, r being a positive integer then,

$$E(X^r) = \int_{-\infty}^{\infty} x^r f(x)\, dx$$

or

$$E(X^r) = \sum_{x} x^r p(x)$$

which is defined as μ_r' (about origin) the r^{th} moment about origin of prob. distribution.

putting $r=1$ i.e. $\phi(X) = X$.

$$E(X) = \sum_{i=1}^{n} x_i p_i = \overline{X} = \mu_1'$$

$$= \text{(first raw moment of } X \text{ about origin } X=0\text{)}$$

If $r=2$ i.e. $\phi(x)=x^2$, $\phi(x_i)=x_i^2$

$\cancel{\phi r}$ $E[\phi(x)] = E[x^2]$

$$= \sum_{i=1}^{n} x_i^2 p_i = \mu_2'$$

(Second order raw moment of r.v X about origin)

If $r=3$ i.e. $\phi(x)=x^3$, $\phi(x_i)=x_i^3$

$$E[\phi(x)] = E(x^3) = \sum_{i=1}^{n} x_i^3 p_i = \mu_3'$$

(3^{rd} order raw mome about origin)

If $r=4$ ie $\phi(x)=x^4$, $\phi(x_i)=x^4$

$E[\phi(x) = E(x^4)$

$$\Rightarrow \sum_{i=1}^{n} x_i^4 p_i = \mu_4'$$

(4^{th} order raw moment about origin.

If $r=0$, $\phi(x) = x^0$, $\phi(x_i) = x_i^0$

$$E[\phi(x)] = E[x^0]$$

$$= \sum_{i=1}^{n} x_i^0 p_i = \sum_{i=1}^{n} p_i = 1$$

So $\boxed{\mu_0' = 1}$

Similarly we can find other moments about origin.

Moments about any point a is defined as :-

We taken $\phi(x) = (x-a)^r$, $\phi(x_i) = (x_i - a$

$r = 1, 2, 3, 4 - -$

So $E[\phi(x)] = E[(x-a)^r]$

$$= \sum_{i=1}^{n} (x_i - a)^r p_i$$

$$= \mu'_{r(x=a)} \rightarrow r^{th}$$ order moment of r.v. X at the point $x = a$

or

$$\boxed{\mu'_{r(x=a)} = E(X-a)^r = \sum_{i=1}^{n} (x_i - a)^r p_i}$$

put $r = 0$

$$\mu'_0 = E(X-a)^0$$

$$= \sum_{i=1}^{n} (x_i - a)^0 p_i$$

$$= \sum_{i=1}^{n} p_i$$

$$\boxed{\mu'_0 = 1}$$

put $r = 1$

$$\mu'_1 = E(X-a)^1 = \sum_{i=1}^{n} (x_i - a) p_i$$

$$= \sum_{i=1}^{n} x_i p_i - \sum_{i=1}^{n} a p_i$$

$$\Rightarrow \quad \bar{X} - a \sum_{i=1}^{n} p_i$$

$[\because \Sigma p_i = 1]$

$$\Rightarrow \quad \boxed{\mu'_{1(x=a)} = \bar{X} - a}$$

putting $r = 2, 3, 4$ we get

$$\mu'_{2(x=a)} = E(x-a)^2 = \sum_{i=1}^{n} (x_i - a)^2 p_i$$

$$\mu'_{3(x=a)} = E(x-a)^3 = \sum_{i=1}^{n} (x_i - a)^3 p_i$$

$$\mu'_{4(x=a)} = E(x-a)^4 = \sum_{i=1}^{n} (x_i - a)^4 p_i$$

Now If We take

$$\phi(x) = (x - \bar{x})^r$$
$$\phi(x_i) = (x_i - \bar{x})^r$$

$$r = 0, 1, 2, 3 \ldots$$

$$E[\phi(x)] = E[(x-\bar{x})^r]$$
$$= \sum_{i=1}^{n} (x_i - \bar{x})^r p_i$$

or

$$\boxed{\begin{aligned}\mu_{r(x=\bar{x})} &= E(x-\bar{x})^r \\ &= \sum_{i=1}^{n} (x_i - \bar{x})^r p_i\end{aligned}}$$

rth order central moment of r.v X about mean put $r=0$

$$\mu_0 = E(x-\overline{x})^0$$

$$= \sum_{i=1}^{n} (x_i-\overline{x})^0 p_i$$

$$= \sum_{i=1}^{n} p_i$$

$$\boxed{\mu_0 = 1}$$ ~~first~~ zero order central moment

put $r=1$

$$\mu_1 = E(x-\overline{x})^1$$

$$= \sum_{i=1}^{n} (x_i-\overline{x}) p_i$$

$$= \sum_{i=1}^{n} x_i p_i - \sum_{i=1}^{n} \overline{x} p_i$$

$$= \overline{x} - \overline{x} \sum_{i=1}^{n} p_i$$

$[\because \Sigma p_i = 1$ total prob. is always 1$]$

$$\Rightarrow \overline{x} - \overline{x}$$

$$\Rightarrow 0$$

$$\boxed{\mu_1 = 0}$$ first order central moment

putting $r=2$

$$\mu_2 = E(x-\bar{x})^2$$

$$= \Sigma (x_i-\bar{x})^2 p_i$$

$$\Rightarrow \Sigma (x_i^2 - 2x_i\bar{x} + \bar{x}^2) p_i$$

$$\Rightarrow \Sigma x_i^2 p_i - 2\bar{x}\,\Sigma x_i p_i + \bar{x}^2 \Sigma p_i$$

$$\Rightarrow \Sigma x_i^2 p_i - 2\bar{x}\cdot\bar{x} + \bar{x}^2$$

$[\because \Sigma x_i p_i = \bar{x} = \mu_1'$

$\Sigma p_i = 1]$

$[\because \Sigma x_i^2 p_i = \mu_2']$

$$\mu_2 \Rightarrow \mu_2' - 2\bar{x}^2 + \bar{x}^2$$

$$\mu_2 = \mu_2' - \bar{x}^2$$

$$\mu_2 = \mu_2' - \mu_1'^2$$

$$\boxed{\mu_2 = \mu_2' - \mu_1'^2 = \sigma^2 = \text{Variance}}$$

Second order central moment.

we can (say)

$$\boxed{\mu_2 = E(x^2) - [E(x)]^2}$$

Similarly :- on putting $r = 3, 4 \ldots$ we get higher order central moments

$$\mu_3 = E(X-\bar{X})^3 = \sum_{i=1}^{n} (x_i - \bar{X})^3 p_i$$

= measures the shape of the curve

$$\mu_4 = E(X-\bar{X})^4 = \sum_{i=1}^{n} (x_i - \bar{X})^4 p_i$$

= measures the height of the curve.

Properties of expectation :-

① If $\phi(x) = ax + b$ where a & b are constants

then

$$E[\phi(x)] = E[ax+b]$$

$$\Rightarrow \sum_{i=1}^{n} (a x_i + b) p_i$$

$$\Rightarrow a \sum_{i=1}^{n} a x_i p_i + b \sum_{i=1}^{n} p_i$$

$[\because \Sigma p_i = 1]$

$$\boxed{E[ax+b] = aE(x)+b}$$

If $a = 0$

then $E[b] = b$

i.e. $\boxed{E[\text{Constant}] = \text{Constant}}$

If $b = 0$

then

$$\boxed{E[ax] = aE(x)}$$

ex → find $E[2x]$ if $\bar{x} = 5$

then $E[2x] = 2 \times E(x)$

$= 2 \times \bar{x} = 2 \times 5$

$= 10$

(2) If $x \geq 0$ then $E(x) \geq 0$

(5) If x & y are two r.v.s i.e. $y \leq x$

then

$$E(y) \leq E(x)$$

(6) $|E(x)| \leq E|x|$

(7) $x_1, x_2, \ldots, x_n$ be r.v and if $a_1, a_2, \ldots, a_n$ are constant

$$E(\Sigma a_i x_i) = \sum_{i=1}^{n} a_i E(X_i)$$

(7) Additive property or Addition theorem or Addition law of expectation :-

Let X & Y be two random variables then, we have

$$E(X+Y) = E(X) + E(Y)$$

i.e. expectation of sum is equal to sum of expectations provided expectations in both sides exist

Discrete case

Let r.v. X takes m values as $x_1, x_2 - - x_m$ with probablites $p_1, p_2 - - p_m$ respectively where $\sum_{i=1}^{m} p_i = 1$

Let r.v. Y takes n values as $y_1, y_2 - - y_n$ with probablities $p'_1, p'_2 - - p'_n$ respectively where $\sum_{j=1}^{n} p'_j = 1$

therefor by definition of mathematical expectation we have

$$E(X) = \sum_{i=1}^{m} x_i p_i$$

&

$$E(Y) = \sum_{j=1}^{n} y_j p_j'$$

Now for L.H.S. sum X+Y is also a r.v. & we takes mn values as $(x_i + y_j)$ with joint prob. as $p_{ij} = P(X = x_i \cap Y = y_j)$

$$= p(x_i, y_j)$$

$$; i = 1, 2 \ldots m$$
$$j = 1, 2 \ldots n$$

where $\sum_{i=1}^{m} \sum_{j=1}^{n} p_{ij} = 1$

Now by def^n of expectation we have

$$E(X+Y) = \sum_{i=1}^{m} \sum_{j=1}^{n} (x_i + y_j) p_{ij}$$

$$= \sum_{i=1}^{m} \sum_{j=1}^{n} (x_i p_{ij} + y_j p_{ij})$$

$$= \sum_{i=1}^{m} \sum_{j=1}^{n} x_i p_{ij} + \sum_{i=1}^{m} \sum_{j=1}^{n} y_j p_{ij}$$

$$= \sum_{i=1}^{m} x_i \sum_{j=1}^{n} p_{ij} + \sum_{j=1}^{n} y_j \sum_{i=1}^{m} p_{ij}$$

$$\Rightarrow \sum_{i=1}^{m} x_i\, p(x_i) + \sum_{j=1}^{n} y_j\, p(y_j)$$

(where $p(x_i) = \sum_{j=1}^{m} p(x_i; y_j)$

$p_i = \sum_{j=1}^{n} p_{ij}$

& $p(y_j) = \sum_{i=1}^{m} p(x_i, y_j)$

$p_j' = \sum_{i=1}^{m} p_{ij}$

are the marginal probablites).

$$\Rightarrow \sum_{i=1}^{m} x_i\, p_i + \sum_{j=1}^{n} y_j\, p_j'$$

$$\Rightarrow E(X) + E(Y)$$

$= RHS$.

so $E(X+Y) = E(X) + E(Y)$

hence proved.

Generalization :-

Let $X_1, X_2 - - X_k$ be k random variables then

$$E(X_1+X_2+ - - +X_n) = E(X_1)+E(X_2)+ - - \pm E(X_n)$$

or

$$E\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} E(X_i)$$

provided expectation exist both sides.

Proof: we know that for two r.v.s. or for $k=2$

$$E(X_1+X_2) = E(X_1) + E(X_2)$$

So $E(X_1+X_2+X_3+ - - + X_n) = E(X_1+Y_1)$

where $Y_1 = X_2+X_3+ - \pm X_k$

$= E(X_1)+E(Y_1)$ [$\because$ applying addition law of expectation for two r.v.]

$\Rightarrow E(X_1) + E(X_2 + X_3 + \cdots X_K)$

$\Rightarrow E(X_1) + E(X_2 + Y_2)$

where $Y_2 = X_3 + X_4 + \cdots + X_K$

$\Rightarrow E(X_1) + E(X_2) + E(Y_2)$

$\Rightarrow E(X_1) + E(X_2) + E[X_3 + X_4 + \cdots + X_K)$

and so - on.

$\Rightarrow E(X_1) + E(X_2) + E(X_3) + \cdots \cdots + E(X_K)$

$=$ R.H.S.

Hence proved.

Continous Case

Let X & Y be two Continous random Variables then

$E(X+Y) = E(X) + E(Y)$

L.H.S.

$\cancel{E(X+Y)} = \iint$

L.H.S.

$$E(X+Y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x+y)\, f(x,y)\, dx \cdot dy$$

where $f(x,y)$ is the joint prob. density function of r.v. $(x+y)$

Range of $x = \int_{-\infty}^{\infty} = \int_{R_X}$

Range of $y = \int_{-\infty}^{\infty} = \int_{R_Y}$

by def^n

$$\left.\begin{aligned} E(X) &= \int_{R_X} x \cdot f(x)\, dx = \int_{-\infty}^{\infty} x\, f(x)\, dx \\ E(Y) &= \int_{R_Y} y \cdot g(y)\, dy = \int_{-\infty}^{\infty} y \cdot g(y)\, dy \end{aligned}\right\} (1)$$

where $f(x)$ & $g(y)$ are the prob. density functions of r.v. X & Y respectively

So, $E(X+Y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x+y)\, f(x,y)\, dx\, dy$

$$\Rightarrow \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x\, f(x,y)\,dx\,dy + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y\, f(x,y)\,dx\,dy$$

$$\Rightarrow \int_{-\infty}^{\infty} x \left[\int_{-\infty}^{\infty} f(x,y)\,dy\right] dx + \int_{-\infty}^{\infty} y \left[\int_{-\infty}^{\infty} f(x,y)\,dx\right] dy$$

$$\Rightarrow \int_{-\infty}^{\infty} x\, f(x)\,dx + \int_{-\infty}^{\infty} y\, g(y)\,dy$$

$\left[\because\right.$ where

$$f(x) = \int_{R_y} f(x,y)\,dy$$

$$g(y) = \int_{R_x} f(x,y)\,dx$$

are the marginal density functions $\left.\right]$

from (1)

$\Rightarrow$

$E(X) + E(Y)$

$=$ R.H.S

Hence proved

(8) Multiplication law of mathematical expectation :-

<u>Theorem</u> :- Let X & Y be two independent random variables then

$$E(X \cdot Y) = E(X) \cdot E(Y)$$

i.e. expectation of product = product of expectation.

Proof: Discrete Case :-

Let r.v. X takes m values as $x_1, x_2 \ldots x_m$ with prob. $p_1, p_2 \ldots p_m$ respectively such that $\sum_{i=1}^{m} p_i = 1$.

Let r.v. Y takes n values as $y_1, y_2 \ldots y_n$ with probablities $p_1', p_2' \ldots p_n'$ respectively such that $\sum_{j=1}^{n} p_j' = 1$

by expectation we have

$$\left.\begin{aligned} E(X) &= \sum_{i=1}^{m} x_i p_i \\ E(Y) &= \sum_{j=1}^{n} y_j p_j' \end{aligned}\right\} \quad (1)$$

Now consider L.H.S. XY is also a r.v. taking mn values as $(x_i y_j)$; $i=1, 2 \ldots m$ & $j=1, 2 - - n$.

with ~~prob~~ joint probablity $p_{ij} = P(X=x_i \cap Y=y_j)$

$= p(x_i, y_j)$ that r.v. X takes ith value x_i & r.v. Y taking jth value y_j

and $\sum_{i=1}^{m} \sum_{j=1}^{n} p_{ij} = 1$

then by definition of expectation, for L.H.S. we can write.

$$E(XY) = \sum_{i=1}^{m} \sum_{j=1}^{n} x_i y_j \, p(x_i, y_j)$$

$$= \sum_{i=1}^{m} \sum_{j=1}^{n} x_i y_j \, P(X = x_i \cap Y = y_j)$$

$$= \sum_{i=1}^{m} \sum_{j=1}^{n} x_i y_j \, P(X = x_i) \, P(Y = y_j)$$

[∵ since X & Y are independent

so joint prob. = product of marginal prob]

$$\Rightarrow \sum_{i=1}^{m} \sum_{j=1}^{n} x_i y_j \, p_i \, p_j'$$

$$= \sum_{i=1}^{m} x_i p_i \cdot \sum_{j=1}^{n} y_j p_j'$$ [∵ from 1]

$$= E(X) \cdot E(Y)$$

$$= \text{RHS}$$

So $E(X \cdot Y) = E(X) \cdot E(Y)$

Hence proved

Generalization :- Let $X_1, X_2 \cdots X_k$ be k mutully independent r.v.s then

$$E(X_1 \cdot X_2 \cdot X_3 \cdots X_k) = E(X_1) \cdot E(X_2) \cdots E(X_k)$$

provided expectation exists both sides.

Proof: we are given $X_1, X_2 \cdots X_n$ are mutully independent

Now.

L.H.S. $E(X_1 \cdot X_2 \cdots X_k)$

$= E(X_1 \cdot Y_1)$ where $Y_1 = X_2 X_3 \cdots X_k$

$= E(X_1) \cdot E(Y_1)$ [$\because$ by multiplication law of expectation for two r.v.s]

$\Rightarrow E(X_1) \cdot E(X_2 \cdot X_3 \cdots X_k)$

$\Rightarrow E(X_1) \cdot E(X_2 \cdot Y_2)$

$\Rightarrow E(X_1) \cdot E(X_2) \cdot E(Y_2)$

$\Rightarrow E(X_1) \cdot E(X_2) \cdot E(X_3 \cdot X_4 \cdot X_5 \cdot \ldots \cdot X_k)$
and so on

$\Rightarrow E(X_1) \cdot E(X_2) \cdot E(X_3) \cdot E(X_4) \cdot \ldots \cdot E(X_k)$

~~so~~ $= RHS$

$E(X_1 \cdot X_2 \cdot \ldots X_k) = E(X_1) \cdot E(X_2) \cdot \ldots E(X_K)$

Hence proved

Continous Case:—

Let X & Y be two continous independent random variables.

$$E(X \cdot Y) = E(X) \cdot E(Y)$$

L.H.S.

$$E(X \cdot Y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x_i y_j) f(x, y) \, dx \, dy$$

Since X & Y are independent so joint prob. = product of marginal probablities.

So $f(x, y) = f(x) \cdot g(y)$

So $E(X \cdot Y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x_i y_j f(x) g(y) \, dx \, dy$

$= \int_{-\infty}^{\infty} x_i f(x) \, dx \cdot \int_{-\infty}^{\infty} y_j g(y) \, dy$

$= E(X) \cdot E(Y)$

$=$ RHS

Hence proved

☆ Expectation & variance of Linear Combinations of Variables :—

Let X & Y be two r.v.s then $U = aX + bY$ —(1) is a linear combination of r.v. X & Y where a & b are constants

The mean of U is given by.

$E(U) = E(aX + bY)$

$= E(ax) + E(by)$ [∵ addition theorem]

$E(U) = aE(X) + bE(Y)$

or $\boxed{\bar{U} = a\bar{X} + b\bar{Y}}$ ← mean of r.v. U — ②

The variance of r.v. U is given by

$$\sigma_U^2 = E[U - \bar{U}]^2$$

from eqn ① & ②

$$U - \bar{U} = (ax + by) - (a\bar{x} + b\bar{y})$$

$$= a(x - \bar{x}) + b(y - \bar{y})$$

$$\sigma_U^2 = E[U - \bar{U}]^2$$

$$\Rightarrow \sigma_U^2 = E[a(x - \bar{x}) + b(y - \bar{y})]^2$$

$$\Rightarrow E[a^2(x - \bar{x})^2 + b^2(y - \bar{y})^2 + 2ab(x - \bar{x})(y - \bar{y})]$$

$$\Rightarrow a^2 E(x - \bar{x})^2 + b^2 E(y - \bar{y})^2 + 2abE(x - \bar{x})(y - \bar{y})$$

$$\Rightarrow a^2 \sigma_x^2 + b^2 \sigma_y^2 + 2ab\,\sigma_{xy}$$

SQ.

$$V(ax+by) = V(U) = a^2 V(x) + b^2 V(y) + 2ab\,Cov(x,y) \quad \text{—— (1)}$$

where

$V(x)$ = Var. of r.v. X

$V(y)$ = Var. of r.v. y

$cov(x,y)$ = Covariance of r.v. X & Y

In (1) if we put $a=1$ & $b=1$ both sides we get

$$V(x+y) = 1^2 V(x) + 1^2 V(y) + 2\cdot1\cdot1\,Cov(x,y)$$

$$\boxed{V(x+y) = V(x) + V(y) + 2cov(x,y)}$$

if $a=1$ $b=-1$

$$V(x-y) = 1^2 V(x) + (-1)^2 V(y) + 2\cdot1\cdot(-1)Cov(x,y)$$

$$\boxed{V(x-y) = V(x) + V(y) - 2cov(x,y)}$$

Question for x, y being independent

$U = 2x + 3y$ $\quad \bar{x} = 3$ $\quad \sigma_x^2 = 4$

$\bar{U} = ?$ $\quad \bar{y} = 4$ $\quad \sigma_y^2 = 5$

$\sigma_U^2 = ?$

$\bar{U} = 2\bar{X} + 3\bar{Y}$

$= 2 \times 3 + 3 \times 4$

$\bar{U} = 18$

$\sigma_U^2 = V(2X + 3Y)$

$= 2^2 V(X) + 3^2 V(Y) + 2^{\times 2 \times 3} \mathrm{cov}(X, Y)$

$\Rightarrow 4V(X) + 9V(Y) + 12\,\mathrm{cov}(X, Y)$

since X & Y are independent
so $\mathrm{cov}(X, Y) = 0$

so $\sigma_U^2 = 4V(X) + 9V(Y) + 0$

$\sigma_U^2 = 4 \times 4 + 9 \times 5$

$= 16 + 45$

$= 61$

$\boxed{\text{covariance of two independent r.v.s} = 0}$

Covariance:—

It is a measure of linear dependence betⁿ two random variables.

Let X & Y be two r. v. s The covariance betⁿ them is defined as

$$Cov.(X,Y) = E\left[\{X-E(X)\}\{Y-E(Y)\}\right]$$

= expectation of product of deviations of r. v. s X & Y from their mean

$$\left[\because E(X)=\bar{X},\ E(Y)=\bar{Y}\right]$$

$$= E\left[(X-\bar{X})(Y-\bar{Y})\right]$$

$$= E\left[XY - X\bar{Y} - \bar{X}Y + \bar{X}\bar{Y}\right]$$

$$= E(XY) - E(X\bar{Y}) - E(\bar{X}Y) + E(\bar{X}\bar{Y})$$

$$= E(XY) - \bar{Y}E(X) - \bar{X}E(Y) + \bar{X}\bar{Y}$$

$$= E(XY) - \bar{X}\bar{Y} - \cancel{\bar{X}\bar{Y}} + \cancel{\bar{X}\bar{Y}}$$

$$= E(XY) - \bar{X}\bar{Y}$$

So $\boxed{\text{cov}(X,Y) = E(XY) - E(X).E(Y)}$

It is the simple form of Covariance.

Note: 1 when r.v.s X & Y are independent then we have

$$E(XY) = E(X) \cdot E(Y)$$ [∵ by multiplication law of expectation]

So

$$\text{cov}(X,Y) = E(X \cdot Y) - E(X) \cdot E(Y)$$

$$= E(X) \cdot E(Y) - E(X) \cdot E(Y)$$
$$= 0$$

~~But its~~ So If two r.v.s X & Y are independent their covariance = 0. But its converse ~~it~~ is not true i.e. if $\text{cov}(X,Y) = 0$ it does not say that X & Y are independent, but they will be linearly independent.

So when X & Y are independent.

$$V(X+Y) = V(X)+V(Y)$$

$$V(X-Y) = V(X)+V(Y)$$

~~Generalization for k random variables for lines~~

~~Let $X_1, X_2 - - X_k$ be k random variables then~~

★ Generalization for k random variables for linear combination - their mean & variance.

Let $X_1, X_2 - - X_k$ be k random variables then

$$U = a_1X_1 + a_2X_2 + - - - a_kX_k$$

is called a r.v. of linear combination of r.v.s $X_1, X_2 - X_k$

$$U = \sum_{i=1}^{k} a_i x_i$$

expectation

$$E(U) = E(a_1 x_1 + a_2 x_2 + \cdots + a_k x_k)$$

$$\bar{U} = a_1 E(x_1) + a_2 E(x_2) + \cdots + a_k E(x_k)$$

$$\bar{U} = a_1 \bar{x}_1 + a_2 \bar{x}_2 + \cdots a_k \bar{x}_k$$

$$\bar{U} = \sum_{i=1}^{k} a_i \bar{x}_i$$

variance → The variance of r.v. U is given by

$$\sigma_U^2 = E\left[U - E(U)\right]^2$$

$$= E\left[U - \bar{U}\right]^2$$

$$= E\left[(a_1 x_1 + a_2 x_2 + \cdots + a_k x_k) - (a_1 \bar{x}_1 + a_2 \bar{x}_2 \cdots + a_k \bar{x}_k)\right]^2$$

$$\Rightarrow E\left[a_1(x_1 - \bar{x}_1) + a_2(x_2 - \bar{x}_2) + \cdots + a_k(x_k - \bar{x}_k)\right]^2$$

$$\Rightarrow E\left[a_1^2(x_1-\bar{x})^2+a_2^2(x_2-\bar{x}_2)^2+\cdots+a_k(x_k-\bar{x}_k)^2\right.$$
$$+2a_1a_2(x_1-\bar{x}_1)(x_2-\bar{x}_2)+2a_2a_3(x_1-\bar{x}_1)(x_3-\bar{x}_3)$$
$$\left.+\cdots+2a_{k-1}a_k(x_{k-1}-\bar{x}_{k-1})(x_k-\bar{x}_k)\right]$$

$$\Rightarrow a_1^2E(x_1-\bar{x}_1)^2+a_2^2E(x_2-\bar{x}_2)^2+\cdots+a_k^2E(x_k-x_k)^2$$
$$+2a_1a_2E(x_1-\bar{x}_1)(x_2-\bar{x}_2)+2a_1a_3E(x_1-\bar{x}_1)(x_3-\bar{x}_3)$$
$$+\cdots+2a_{k-1}a_kE(x_{k-1}-\bar{x}_{k-1})(x_k-\bar{x}_k)$$

$$\Rightarrow \sigma_U^2 = a_1^2V(x_1)+a_2^2V(x_2)+\cdots+a_k^2V(x_k)$$
$$+2a_1a_2Cov(x_1,x_2)+2a_1a_3Cov(x_1,x_3)$$
$$+\cdots+2a_{k-1}a_kCov(x_{k-1},x_k)$$

So $\boxed{V\left(\sum_{i=1}^{k}a_ix_i\right)=\sum_{i=1}^{k}a_i^2V(x_i)+2\sum_{i=1}^{k}\sum_{j=1}^{k}a_ia_jCov(x_i,x_j)}$

variance of sum = sum of variance
+ sum of covariances

Note:- If r.v.s $X_1, X_2 - - X_K$ mutually independent then

$$cov(X_i, X_j) = 0 \text{ for } i \neq j = 1, 2 - - k$$

So we have

$$\boxed{V\left(\sum_{i=1}^{k} a_i X_i\right) = \sum_{i=1}^{k} a_i^2 V(X_i)}$$

If $a_1 = a_2 = - - = a_K = 1$

$$V\left(\sum_{i=1}^{k} X_i\right) = \sum_{i=1}^{k} V(X_i)$$