Theory of finite differences

—x—x—x—x—x—x—x—

There are two types of variables. ~~which are~~

i) Argument :- One which is independent is known as argument.

ii) Entry :- The variable which is ~~in~~ dependent on the other variable is known as entry.

Different types of operators:-

i) Identity operator → The Identity Operator 'I' operates just like one (1) i.e.

$$I f(x) = f(x)$$

(ii) Shift operator (E) → If $y = f(x)$ is any function then operation $E f(x)$ implies that give an increment to the value of x in the function. If this increment is of quantity "h" then operation E means means that put $(x+h)$ in the function wherever there is x i.e.

$$E f(x) = f(x+h)$$

$$E^2 f(x) = E(E f(x)) = E(f(x+h))$$

$$= f(x+h+h) = f(x+2h)$$

Similarly

$$E^n f(x) = f(x+nh)$$

& $$E^{-1} f(x) = f(x-h)$$

$$E^{-n} f(x) = f(x-nh)$$

(iii) Forward difference operator (Δ):- when Δ is operated on a function it results as

$$\Delta f(x) = f(x+h) - f(x)$$

$$\Delta f(x) = E f(x) - I f(x)$$

$$\Delta f(x) = (E-I) f(x)$$

$$\boxed{\Delta = E - I}$$

Relation between operator E & Δ

$$\Delta = E - I$$

$$\Delta f(x) = (E-I) f(x)$$

$$\Delta^2 f(x) = \Delta . \Delta f(x)$$
$$= \Delta [f(x+h) - f(x)]$$
$$= \Delta f(x+h) - \Delta f(x)$$
$$= f(x+2h) - f(x+h) - [f(x+h) - f(x)]$$
$$= f(x+2h) - f(x+h) - f(x+h) + f(x)$$
$$= f(x+2h) - 2f(x+h) + f(x)$$
$$= E^2 f(x) - 2E f(x) + f(x)$$
$$= (E^2 - 2E + 1) f(x)$$
$$\Delta^2 f(x) = (E-1)^2 f(x)$$

Similarly

$$\Delta^3 f(x) = (E-1)^3 f(x)$$

$$\vdots$$

$$\Delta^n f(x) = (E-1)^n f(x)$$

Example:- ① find $\Delta f(x)$ when $f(x)=x^2$ & $h=1$

$$\Delta f(x) = f(x+h) - f(x)$$

$$\Delta x^2 = (x+1)^2 - x^2$$

$$= \cancel{x^2} + 2x + 1 - \cancel{x^2}$$

$$= 2x+1 \quad \text{Ans}$$

② find $\Delta^2 f(x)$

$$\Delta^2 f(x) = \Delta \cdot \Delta \cdot f(x)$$

$$= \Delta . (f(x+h) - f(x))$$

$$= \Delta f(x+h) - \Delta f(x)$$

$$= f(x+2h) - f(x+h) - f(x+h) + f(x)$$

$$= f(x+2h) - 2f(x+h) + f(x)$$

On putting values.

$$\Delta^2 . x^2 = (x+2)^2 - 2(x+1)^2 + x^2$$

$$= x^2 + 4x + 4 - 2(x^2 + 2x + 1) + x^2$$

$$= \cancel{x^2} + \cancel{4x} + 4 - \cancel{2x^2} - \cancel{4x} - 2 + \cancel{x^2}$$

$$= 4 - 2$$

$$= 2 \quad \underline{\text{Ans}}$$

4) Backward Difference Operator (∇ nebla) $\rightarrow$

When the operator ∇ is operated $f(x)$ results as

$$\boxed{\nabla f(x) = f(x) - f(x-h)}$$

Relation betn ∇ & E

$$\nabla f(x) = f(x) - f(x-h)$$

$$\nabla f(x) = f(x) - E^{-1} f(x)$$

$$\nabla f(x) = (1 - E^{-1}) f(x)$$

$$\boxed{\nabla = (1 - E^{-1})}$$

$$\nabla^2 f(x) = \nabla . \nabla . f(x)$$

$$= \nabla . [f(x) - f(x-h)]$$

$$= \nabla f(x) - \nabla f(x-h)$$

$$\Rightarrow f(x) - f(x-h) - [f(x-h) - f(x-2h)]$$

$$\Rightarrow f(x) - f(x-h) - f(x-h) + f(x-2h)$$

$$= f(x) - 2f(x-h) + f(x-2h)$$

$$= f(x) - 2E^{-1} f(x) + E^{-2} f(x)$$

$$= (1 - 2E^{-1} + E^{-2}) f(x)$$

$$\boxed{\nabla^2 f(x) = (1 - E^{-1})^2 f(x)}$$

Similarly.

$$\nabla^n f(x) = (1 - E^{-1})^n f(x).$$

5) Differentiation operator (B) → From Taylor's theorem we have (3)

$$f(x+h) = f(x) + \frac{h f'(x)}{\lfloor 1} + \frac{h^2}{2} f''(x) + \cdots$$

$$E f(x) = f(x) + \frac{h B f(x)}{\lfloor 1} + \frac{h^2}{\lfloor 2} B^2 f(x) + \cdots$$

$$E f(x) = f(x) \left[1 + \frac{hB}{\lfloor 1} + \frac{h^2 B^2}{\lfloor 2} + \cdots \right]$$

$$E f(x) = e^{hB} f(x) \qquad \left(\because e^x = 1 + \frac{x}{\lfloor 1} + \frac{x^2}{\lfloor 2} + \cdots\right)$$

$$\boxed{E = e^{hB}}$$

$$\Rightarrow \boxed{1 + \Delta = e^{hB}} \qquad (\because \Delta = E - 1)$$

$$\Rightarrow \log(1+\Delta) = hB$$

$$\Rightarrow B = \frac{1}{h}\left[\Delta - \frac{\Delta^2}{2} + \frac{\Delta^3}{3} - \cdots\right]$$

Example:- $f(x) = x(x-1)(x-2)$ operate forward difference operator Δ sequentially 3 times & $h = 1$

Sol^n.

$$\Delta . f(x) = f(x+h) - f(x)$$

$$= (x+h)(x+h-1)(x+h-2) - x(x-1)(x-2)$$

$$= (x+1)(x+1-1)(x+1-2) - x(x-1)(x-2)$$

$$= (x+1)x(x-1) - x(x-1)(x-2)$$

$$= x(x-1)[(x+1)-(x-2)]$$

$$= x(x-1)(x+1-x+2)$$

$$\Rightarrow x(x-1)3 = 3x(x-1) = 3x^2 - 3x$$

$$\Delta^2 f(x) = \Delta \cdot \Delta f(x)$$
$$= \Delta \cdot (3x^2 - 3x)$$
$$= 3(x+h)^2 - 3(x+h) - 3x^2 + 3x$$
$$= 3(x+1)^2 - 3(x+1) - 3x^2 + 3x$$
$$= 3(x^2+2x+1) - 3x - 3 - 3x^2 + 3x$$
$$= \cancel{3x^2} + 6x + \cancel{3} - \cancel{3x} - \cancel{3} - \cancel{3x^2} + \cancel{3x}$$
$$\Delta^2 f(x) = 6x$$

$$\Delta^3 f(x) = \Delta \cdot \Delta^2 f(x)$$
$$= \Delta \cdot 6x$$
$$= 6(x+h) - 6x$$
$$= 6(x+1) - 6x$$
$$= \cancel{6x} + 6 - \cancel{6x}$$
$$= 6 \quad \underline{\text{Ans.}}$$

Relation between Δ & ∇

— x — x — x — x — x —

$$\Delta f(x) = f(x+h) - f(x) = \nabla f(x+h)$$

So $\Delta f(x) = \nabla E f(x)$

$$\boxed{\Delta = \nabla E}$$

Properties of operators E & Δ (Algebric properties

i) Operators E & Δ are distributive:- Let any function $U(x)$ be the sum of functions $f(x), g(x), p(x)$ - - - So that

$$U(x) = f(x) + g(x) + p(x) + - - - -$$

Then by definition:-

$$E \cdot U(x) = f(x+h) + g(x+h) + \phi(x+h) + \cdots$$

[the interval of difference being h

$$= E \cdot f(x) + E \cdot g(x) + E \cdot \phi(x) + \cdots$$

which shows that E is distributive.

Similarly

$$\Delta\, U(x) = \Delta\{f(x) + g(x) + \phi(x) + \cdots\}$$

$$= [f(x+h) + g(x+h) + \phi(x+h) + \cdots]$$

$$- [f(x) + g(x) + \phi(x) + \cdots]$$

$$= [f(x+h) - f(x)] + [g(x+h) - g(x)] + \cdots$$

$$= \Delta \cdot f(x) + \Delta \cdot g(x) + \Delta \cdot \phi(x)$$

which shows Δ is also distributive.

(ii) E & Δ are commutative with regard to constant

i.e. $$E \cdot C\, U(x) = C\, E \cdot U(x)$$

& $$\Delta \cdot C\, U(x) = \cancel{\Delta}\, C\, \Delta \cdot U(x)$$

where C is some constant

Proof:- Now ~~E.C.Cu~~

$$E \cdot C\, U(x) = C \cdot U(x+h)$$

$$= C \cdot E \cdot U(x)$$

and

$$\Delta\, C\, U(x) = C\, U(x+h) - C \cdot U(x)$$

$$= C\,[U(x+h) - U(x)]$$

$$= C \cdot \Delta\, U(x)$$

which shows E & Δ are commutative w.r.t constant.

(iii) E & Δ obey the laws of indices.

i.e. $$E^m \cdot E^n \cdot U(x) = E^{m+n}\, U(x)$$

& $$\Delta^m \cdot \Delta^n \cdot U(x) = \Delta^{m+n}\, U(x)$$

Proof: $$E^m \cdot E^n \cdot U(x) = E^m \cdot U(x+nh)$$
$$= U(x+nh+mh)$$
$$= U(x+(m+n)h)$$
$$= E^{m+n}\, U(x)$$

and

$$\Delta^m \cdot \Delta^n\, U(x) = (\Delta \cdot \Delta \cdots m \text{ times})(\Delta \cdot \Delta \cdots n \text{ times})\, U(x)$$
$$= (\Delta \cdot \Delta \cdots (m+n) \text{ times})\, U(x)$$
$$= \Delta^{m+n}\, U(x)$$

Operator E & Δ can be regarded as algebric quantities, ~~but~~

(IV) E & Δ are not commutative w.r.t. variables

If $$U(x) = f(x) \cdot g(x)$$
$$E \cdot U(x) \neq f(x)\, E\, g(x)$$
$$\Delta\, U(x) \neq f(x)\, \Delta\, g(x)$$

(V) $\cancel{E^{-n} f(x) = f(x-nh)}$ ⑤

(V) $E^2 f(x) \neq \{E f(x)\}^2$

Proof: R.H.S.

$[E f(x)]^2 = [f(x+h)]^2$

$L.H.S. = E^2 f(x) = f(x+2h)$

$L.H.S \neq R.H.S.$

(vi) If $\Delta f(x) = 0$ then it does not imply that either $\Delta = 0$ or $f(x) = 0$

(vii) Operators E & Δ cannot stand without the operand.

(viii) The finite difference of a product of two functions

$\Delta\{f(x) \cdot g(x)\} = f(x+h)\,\Delta g(x) + g(x)\,\Delta f(x)$

Proof L.H.S.

$\Delta\{f(x) \cdot g(x)\} = f(x+h)\,g(x+h) - f(x) \cdot g(x)$

$\Rightarrow f(x+h)\,g(x+h) - f(x+h)\,g(x) + f(x+h)\,g(x) - f(x) \cdot g(x)$

$\Rightarrow f(x+h)[g(x+h) - g(x)] + g(x)[f(x+h) - f(x)]$

$= f(x+h)\,\Delta g(x) + g(x)\,\Delta f(x)$

$= R.H.S.$

Hence proved

(ix) The finite difference of a quotient of two functions:

$$\Delta\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\,\Delta f(x) - f(x)\,\Delta g(x)}{g(x+h)\,g(x)}$$

Proof: L.H.S. $\Delta\left[\frac{f(x)}{g(x)}\right]$

$$= \frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}$$

$$= \frac{f(x+h)\cdot g(x) - g(x+h)\cdot f(x)}{g(x+h)\,g(x)}$$

$$= \frac{f(x+h)\cdot g(x) - g(x+h)f(x) + f(x)g(x) - f(x)g(x)}{g(x+h)\,g(x)}$$

$$= \frac{f(x+h)\cdot g(x) - f(x)g(x) - g(x+h)f(x) + f(x)g(x)}{g(x+h)\,g(x)}$$

$$= \frac{g(x)[f(x+h) - f(x)] - f(x)[g(x+h) - g(x)]}{g(x+h)\,g(x)}$$

$$= \frac{g(x)\,\Delta f(x) - f(x)\,\Delta g(x)}{g(x+h)\,g(x)}$$

$=$ R.H.S.

Hence proved

Difference Table

1) Forward difference table

x	$y=f(x)$	Δy	$\Delta^2 y$	$\Delta^3 y$
x_0	y_0	$\Delta y_0 = y_1 - y_0$	$\Delta^2 y_0 = \Delta y_1 - \Delta y_0$	$\Delta^3 y_0 = \Delta^2 y_1 - \Delta^2 y_0$
$x_1 = x_0 + h$	y_1	$\Delta y_1 = y_2 - y_1$	$\Delta^2 y_1 = \Delta y_2 - \Delta y_1$	$\Delta^3 y_1 = \Delta^2 y_2 - \Delta^2 y_1$
$x_2 = x_0 + 2h$	y_2	$\Delta y_2 = y_3 - y_2$	$\Delta^2 y_2 = \Delta y_3 - \Delta y_2$	$\Delta^3 y_2 = \Delta^2 y_3 - \Delta^2 y_2$
$x_3 = x_0 + 3h$	y_3	$\Delta y_3 = y_4 - y_3$	$\Delta^2 y_3 = \Delta y_4 - \Delta y_3$	
$x_4 = x_0 + 4h$	y_4	$\Delta y_4 = y_5 - y_4$		
$x_5 = x_0 + 5h$	y_5			

2) Backward Difference table

x	$y=f(x)$	∇y	$\nabla^2 y$	$\nabla^3 y$
x_0	y_0			
$x_1 = x_0 + h$	y_1	$\nabla y_1 = y_1 - y_0$		
$x_2 = x_0 + 2h$	y_2	$\nabla y_2 = y_2 - y_1$	$\nabla^2 y_2 = \nabla y_2 - \nabla y_1$	
$x_3 = x_0 + 3h$	y_3	$\nabla y_3 = y_3 - y_2$	$\nabla^2 y_3 = \nabla y_3 - \nabla y_2$	$\nabla^3 y_3 = \nabla^2 y_3 - \nabla^2 y_2$
$x_4 = x_0 + 4h$	y_4	$\nabla y_4 = y_4 - y_3$	$\nabla^2 y_4 = \nabla y_4 - \nabla y_3$	$\nabla^3 y_4 = \nabla^2 y_4 - \nabla^2 y_3$
$x_5 = x_0 + 5h$	y_5	$\nabla y_5 = y_5 - y_4$	$\nabla^2 y_5 = \nabla y_5 - \nabla y_4$	$\nabla^3 y_5 = \nabla^2 y_5 - \nabla^2 y_4$