Moment Generating function

Def^n:→ The Moment Generating function of a prob. dist^n function of a r.v. x about origin is defined as

$$M_x(t) = E[e^{tx}]$$

where t is a real number (Value)

$x=$ becz taking particular value $\quad = \sum_{x=0}^{\infty} e^{tx} p(x)$ if x is a discrete r.v.

$$= \int_{-\infty}^{\infty} e^{tx} f(x)\,dx \text{ if } x \text{ is Conti. r.V.}$$

Properties: (i) $M_x(t) = \sum_{r=0}^{\infty} \frac{t^r}{\lfloor r} \mu_r'$

proof:- $M_x(t) = E[e^{tx}]$

$$= E\left[1 + \frac{tx}{\lfloor 1} + \frac{(tx)^2}{\lfloor 2} + \dots + \frac{t^r x^r}{\lfloor r}\right]$$

$$= 1 + t E(x) + \frac{t^2}{\lfloor 2} E(x^2) + \text{———}$$

$t^1 E(x-0)$ $\quad$ $E(x-0)^2$

$\mu_1' = x'$

$\mu_1 = 0$ $\quad = 1 + t\mu_1' + \frac{t^2}{\lfloor 2}\mu_2' + \frac{t^3}{\lfloor 3}\mu_3' \dots + \frac{t^r}{\lfloor r}\mu_r'$ ——— (1)

$\mu_1' =$ raw Moment

$\mu_1 =$ central Moment $\quad = \sum_{r=0}^{\infty} \frac{t^r}{\lfloor r} \mu_r'$ ——— (1.1)

Corollary (i) $\mu_r' =$ coeff. of $\frac{t^r}{\lfloor r}$ in $M_x(t)$

for $r = 1, 2, 3 \ldots$

Corollary (ii)

$$\mu_r' = \left[\frac{d^r}{dt^r} M_x(t)\right]_{t=0}$$

proof:- Differentiation (1) w.r.t. 't'

$$\frac{d}{dt} M_x(t) = \frac{d}{dt}\left[1 + t\mu_1' + \frac{t^2}{\lfloor 2}\mu_2' + \ldots\right]$$

$$= 0 + \mu_1' + \frac{2t}{\lfloor 2}\mu_2' + \frac{3t^2}{\lfloor 3}\mu_3' + \ldots \quad (1.2)$$

as $\left[\frac{d}{dt} M_x(t)\right]_{t=0} = \mu_1'$

diff. (1.2) w.r.t. 't'

$$\frac{d^2}{dt^2} M_x(t) = \mu_2' + \frac{6t}{\lfloor 3}\mu_3' + \text{terms with higher power}$$

$$\left[\frac{d^2}{dt^2} M_x(t)\right]_{t=0} = \mu_2' + 0 \text{ so on}$$

Similarily $= \left[\frac{d^r}{dt^r} M_x(t)\right]_{t=0} = \mu_r'$

* Effect of change of Scale and Origin on M.G.F.

If $y = ax + b$; then

$$M_y(t) = e^{bt} M_x(at) \quad t\text{-}\mathrm{const.}$$

proof :→ $M_y(t) = M_{ax+b}(t)$

$$= E\left[e^{t(ax+b)}\right]$$

$$= E\left[e^{tax} \cdot e^{bt}\right]$$

$$= E\left[e^{tax}\right] \cdot E\left[e^{bt}\right]$$

$$= e^{bt} \cdot E\left[e^{tax}\right]$$

$$= e^{bt} M_x(at)$$

$\left[\begin{array}{l} E(Kx) \\ = K E(x) \\ E(K) = K \end{array}\right.$

* additive property :→ If X and Y are two independent random Variables then

$$M_{x+y}(t) = M_x(t) \cdot M_y(t)$$

Proof:- $M_{x+y}(t) = E\left[e^{t(x+y)}\right]$

$$= E\left[e^{tx} \cdot e^{ty}\right]$$

$$= E(e^{tx}) \cdot E(e^{ty}) = M_x(t) M_y(t)$$

In General if $x_1, x_2 \ldots x_n$ are indpt r.v. then

$$M_{x_1+x_2+\ldots+x_n}(t) = M_{x_1}(t) \cdot M_{x_2}(t) \ldots M_{x_n}(t)$$

~~i.e $M_{\mu}\Sigma$~~

i.e $M_{\sum_{i=1}^{n} x_i}(t) = \prod_{i=1}^{n} M_{x_i}(t)$

Note :- (i) If $M_x(t) = M_y(t)$

then $P(x) = P(y)$

Provided the m.g.f.'s exists by uniqueness theorem any dist^n function has only one m.g.f. or any prob dist^n function has unique m.g.f.

Note = Moments Generating function of Cauchy dist^n doesn't exist.

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Q.1 Show that (i) μ_r = coeff. of $\frac{t^r}{\lfloor r}$ in $M_{x-\mu}(t)$

(ii) $\mu_r = \left[\frac{d^r}{dt^r} M_{x-\mu}(t) \right]_{t=0}$

Proof :→ The central moment of a r.v. x is defined as,

(i) $M_{x-\mu}(t) = E\left[e^{t(x-\mu)} \right]$

$= \sum_{x=0}^{\infty} e^{t(x-\mu)} \cdot p(x)$

of r.v. x is discrete

$$= \int_{-\infty}^{\infty} e^{t(x-\mu)} f(x)\,dx$$

r.v. X is continuous

$$= E\left[e^{t(x-\mu)}\right]$$

$$= E\left[1 + t(x-\mu) + \frac{t^2}{\lfloor 2}(x-\mu)^2 + \frac{t^3}{\lfloor 3}(x-\mu)^3 + \dots + \frac{t^r}{\lfloor r}(x-\mu)^r\right]$$

$$= 1 + t\,E(x-\mu) + \frac{t^2}{\lfloor 2}E(x-\mu)^2 + \dots + \frac{t^r}{\lfloor r}E(x-\mu)^r$$

$$= 1 + t\mu_1 + \frac{t^2}{\lfloor 2}\mu_2 + \dots + \frac{t^r}{\lfloor r}\mu_r$$

i.e. μ_r = coeff. of $\frac{t^r}{\lfloor r}$ $M_{x-\mu}(t)$

(II) $$\mu_r = \left[\frac{d^r}{dt^r} M_{x-\mu}(t)\right]$$

$$M_{x-\mu}(t) = 1 + t\mu_1 + \frac{t^2}{\lfloor 2}\mu_2 + \dots \quad (1.1)$$

diff. (1.1) w.r.t. 't' and

$$\frac{d}{dt} M_{x-\mu}(t) = 0 + \mu_1 + \frac{2t}{\lfloor 2}\mu_2 + \dots \quad (1.2)$$

or $$\left[\frac{d}{dt} M_{x-\mu}(t)\right]_{t=0} = \mu_1$$

again diff. (1.2) w.r.t. 't'

$$\frac{d^2}{dt^2} M_{x-\mu}(t) = \mu_2 + \frac{6t}{\lfloor 3} \mu_3 t$$

$$\left[\frac{d^2}{dt^2} M_{x-\mu}(t)\right]_{t=0} = \mu_2$$

Similarily $\left[\frac{d^r}{dt^r} M_{x-\mu}(t)\right]_{t=0} = \mu_r$

* Generating function :→ Generating function of a ~~seq~~ sequence $\{a_0, a_1, a_2 \ldots\} = \{a_i\}$ of a real No. is defined and denoted by

$$A(s) = a_0 + a_1 s + a_2 s^2 + a_3 s^3 + \ldots + \infty$$

$$= \sum_{i=0}^{\infty} a_i s^i$$

provided $A(s)$ ~~in~~ is convergent in $|s|$

<u>Example</u> :→ find the G.F. of sequence $\{1, 1, 1 \ldots\} = a_i$

here $a_i = 1$

then $A(s) = 1 + s + s^2 + s^3 + \ldots$

$$= \frac{1}{1-s} \quad \text{for } |s| < 1$$

$$\left[1 + x + x^2 + \ldots = (1-x)^{-1}\right]$$

Questn - find Generating function of
{0, 0, 1, 1, 1 ...}

Ans:- → Generating function is define as

$$A(S) = a_0 + a_1 S + a_2 S^2 + a_3 S^3 + \dots$$

here $a_0 = 0$, $a_1 = 0$, $a_2 = 1$, $a_3 = 1$, $a_i = 1$ $i \geq 2$

Then $A(S) = 0 + S^2 + S^3 + \dots$

$$= S^2 (1 + S + S^2 + \dots)$$

$$= \frac{S^2}{1-S}$$

Questn :→ find the Generating function of $\{a_i\}$ where $a_i = 1/\underline{i}$

Ans:- $$A(S) = \sum_{i=0}^{\infty} a_i s^i = \sum_{i=0}^{\infty} \frac{s^i}{\underline{i}}$$

$$= 1 + s + \frac{s^2}{\underline{2}} + \frac{s^3}{\underline{3}} + \dots$$

$$= e^s$$

Questn ③ find the Generating functn of $\{a_i\}$ where $a_i = {}^nC_i$

Ans:- $$A(S) = \sum_{i=0}^{\infty} a_i s^i$$

$$= \sum_{i=0}^{\infty} {}^nC_i s^i$$

G.F. में $\{a_i\}$ कोई भी real no ...
but P.G.F. में particular prob funct^n जैसा होगा)
prob dist^n होगा उसका prob funct^n होगा।

$$= 1 + {}^nC_1 s + {}^nC_2 s^2 + {}^nC_3 s^3 + \ldots$$

$$= (1-s)^{-n}$$

Quest^n (4) find Generating function of $\{a_i\}$ where

$$a_i = \text{Prob.}[\,i = \text{No. on a dice drawn once}\,]$$

Ans: $\to$ $a_i = \text{Prob.}[\,i = 1, 2, 3, 4, 5, 6\,]$

$$\{a_i\} = \left\{\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}\right\}$$

$$A(s) = \frac{1}{6} + \frac{1}{6}s + \frac{1}{6}s^2 + \frac{1}{6}s^3 + \frac{1}{6}s^4 + \frac{1}{6}s^5$$

$$= \frac{1}{6}\{1 + s + s^2 + \ldots + s^5\}$$

$$= \cancel{\frac{1}{6}(1-s)^{-1}} = \frac{1}{6}\left\{\frac{1-s^6}{1-s}\right\} \qquad |s| < 1$$

Probability Generating function (P.G.f.) $\Rightarrow$

P.G.F. of an integral value discrete random var X is defined as a generating funct^n of a sequence $\{p_x\}$ where $p_x = P(X = x)$ denoted by

$$P_X(s) = \sum_{x=0}^{\infty} p_x s^x \quad \text{where } p_x = P(X = x)$$