Unit IV

Quadratic form :- A homogeneous polynomial of the type.

$$Q = \cancel{Q(x_1, x_2)} \cdot Q(x, x)$$

$$= (a_{11}x_1^2 + a_{12}x_1x_2 + a_{13}x_1x_3 + \cdots$$

$$\cdots + a_{1n}x_1x_n)$$

$$+ (a_{21}x_1x_2 + a_{22}x_2^2 + a_{23}x_2x_3 + \cdots$$

$$+ x_{2n}x_2x_n) + \cdots$$

$$\cdots$$

$$+ \cancel{(a_{n1}x_1x_n + a_{n2}x_2x_n}$$

$$+ (a_{n1}x_1x_n + a_{n2}x_2x_n + \cdots + a_{nn}x_n^2)$$

is called a Quadratic Polynomial in the variable $x_1, x_2, \ldots, x_n$. when all the coefft. are real then the quadratic form is called a real Quadratic form. Further variables $x_1, x_2, \ldots, x_n$ are also consider to be real.

Example :- Consider Quadratic form.

$$Q = x_1^2 + 2x_2^2 - 7x_3^2 - 4x_1x_2 + 8x_2x_3$$

$$Q = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 1 & -2 & 0 \\ -2 & 2 & 4 \\ 0 & 4 & -7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Here the coeffⁿ matrix is

$$A = \begin{bmatrix} 1 & -2 & 0 \\ -2 & 2 & 4 \\ 0 & 4 & -7 \end{bmatrix}$$

matrix $X = [x_1 \quad x_2 \quad x_3]^T$

A general quadratic form involving n variable can be put in the form

$$Q = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_i x_j$$

$$= X^T A X$$

where $= \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$

is the coeffⁿ matrix of the Quadratic form. The square matrix A of order n is called the matrix of the quadratic form (Q.f.)

Its determinent is given by

det $A = |A|$. is called the discriminant of quadratic form. The rank of the matrix A is rank of the Q.f.

& If $|A| = 0$. then the Q.f. is a

singular Q.F. otherwise it is a non-singular Q.f. further if the Q.f. has the elements of the coefficient matrix as complex no.'s & so are the set of variables $x_1, x_2, \ldots, x_n$ also complex then the Q.F. is called a Hermitian Q.F.

NOTE :- The coefficient matrix of the Q.f.

$$ax^2 + by^2 + cz^2 + 2hxy + 2fyz + 2gxz = 0$$

is given by

$$A = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$$

Quadratic form :-

By effecting the linear transformation

$X = BY$ or $Y = B^{-1}X$ (provided that B^{-1} exists)

where $B = [b_{ij}]$ is a square matrix of order n.

$$X = [x_1, x_2, \ldots, x_n]^T \text{ \& } Y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

By using the elementary transformation defined above, we can express the quadratic form

$$X^T A X = (BY)^T A (BY)$$
$$= (Y^T B^T) A (BY)$$
$$= Y^T B^T A B Y$$
$$= Y^T C Y$$

where C is the matrix given by

$$C = B^T A B$$

The new quadratic form $Y^T C Y$ is a quadratic form with new set of variables $y_1, y_2, \ldots, y_n$.

Canonical form :- If by any non-singular linear transformation a real quadratic form can be expressed as a sum or difference of the square of the new variables then the new quadratic expression is called Canonical form of the given Q.f.

If the non-singular linear transformation $X = PY$ reduces the given quadratic form

$$ax^2 + by^2 + cz^2 + 2hxy + 2fyz + 2gxz$$ to

$$k_1 y_1^2 + k_2 y_2^2 + k_3 y_3^2$$

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then $K_1 y_1^2 + K_2 y_2^2 + K_3 y_3^2$ is called the Canonical form of the given Q.f.

Theorem :— Every Quadratic form can be reduces to a form containing only square terms (also known as canonical form) by a non-singular linear transformation.

proof :— Let $X^T A X$ be the Q.F. with A as square matrix of order n by using elementary transformation we can reduce A to a form which is of the type. $B^T A B = \Delta$

where B is an orthogonal matrix & Δ is a diagonal matrix. We choose the matrix B for constructing the non-singular transformation given by

$$Y = B^{-1} X \quad \text{or} \quad X = BY$$

Now

$$X^T A X = (BY)^T A (BY)$$

$$= Y^T B^T A B Y$$

$$= Y^T \Delta Y$$

[Y^T is row vector & Y is column]

$$= Y^T \text{ diagonal } [\lambda_1, \lambda_2, \ldots, \lambda_n] Y$$

$$= \lambda_1 y_1^2 + \lambda_2 y_2^2 + \ldots + \lambda_n y_n^2$$

where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the elements of the diagonal matrix Δ

this prove the theorem.

NOTE :— When a Q.F. to reduce to sum of square or canonical form in terms of

$\lambda_1 y_1^2 + \lambda_2 y_2^2 + \ldots + \lambda_n y_n^2$. then the no.'s of positive λ's are called the index, the total no's of positive λ's is called the rank of the given Q.F.. also total no's of positive λ's (−) total no. of negative λ's is called the signature of the Q.F.

Eg: Reduce the Q.f. given by

$$Q = x_1^2 + 2x_2^2 - 7x_3^2 - 4x_1x_2 + 8x_1x_3$$

to sum of squares or to canonical form. find the rank, index & signature.

Soln:— The matrix of the Quadratic form is

$$A = \begin{bmatrix} 1 & -2 & 4 \\ -2 & 2 & 0 \\ 4 & 0 & -7 \end{bmatrix}$$

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Let us write

$$\begin{bmatrix} 1 & -2 & 4 \\ -2 & 2 & 0 \\ 4 & 0 & -7 \end{bmatrix}_{3\times 3} = \overset{B^T}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \times \overset{A}{A} \overset{B}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

$\cancel{3\times3}$

$(R_2 + 2R_1), (R_3 + 2R_2)$

$$\begin{bmatrix} 1 & -2 & 4 \\ 0 & -2 & 8 \\ 0 & 4 & -7 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$(C_2 + 2C_1, \quad C_3 + 2C_2)$

Then

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 4 \\ 0 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_3 + 2R_2$

Then

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 4 \\ 0 & 0 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 4 & 4 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

$C_3 + 2C_2$

Then $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 4 & 4 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$

Put $B^T = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 4 & 4 & 1 \end{bmatrix}$

using the transformation

$X = BY$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

this gives

$$\left.\begin{aligned} x_1 &= y_1 + 2y_2 + 4y_3 \\ x_2 &= y_2 + 4y_3 \\ x_3 &= y_3 \end{aligned}\right\} - (1)$$

the transformation (1) reduce to given Quadratic form to

$$Y^T \operatorname{diag}[1 \quad -2 \quad 9] \, Y$$

$$= y_1^2 - 2y + 9y_3^2$$

This is the canonical form

Here index $= 2$, rank $= \cancel{3}\ 2$ &
signature $= 1$

Ex. Reduce to cannonical form the following:

(i) $x_1x_2 + x_2x_3 + x_3x_1$

(ii) $yz - 2zx + xy$

(iii) $6x_1^2 - 3x_2^2 + 14x_3^2 + 4x_2x_3 + 18x_3x_1 + 4x_1x_2$

find also the rank index & signature

Solⁿ The matrix of the Q.F.

$x_1x_2 + x_2x_3 + x_3x_1$ is

$$A = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}$$

Let us write

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_1 + R_2$, then

$$\begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & \cancel{1} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$(R_2-R_1 \cdot R_3-R_1)$, then

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & -1 & -1 \\ 0 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 0 \\ -1 & -1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}$$

$(C_2-C_1, \ C_3-2C_1)$, then

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 0 \\ -1 & -1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & 2 \\ -1 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}$$

$(-C_2+C_3)$ i.e. (C_3-C_2)

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & -0 \\ -1 & 0 & 0 \\ -1 & -1 & 2 \end{bmatrix} A \begin{bmatrix} 1 & +1 & -1 \\ -1 & 0 & 0 \\ -1 & -1 & 2 \end{bmatrix}$$

Put $B^T = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & -1 & 2 \end{bmatrix}$

using the transformation

$X = BY$.

$$\begin{bmatrix} x_1 \\ x_2 \\ \end{bmatrix} = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \end{bmatrix}$$

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this gives

$$x_1 = y_1 - y_2 - y_3$$
$$x_2 = y_2 - y_3$$
$$x_3 = 2y_3$$
(1)

the transformation (1) reduces to the given Q.F. to

$$Y^T \text{diag.}(1, -1, -2)\, Y$$
$$= Y_1^2 - y_2^2 - 2y_3^2$$

this is the canonical form
there index is 2, rank = 3, &
signature = −1

Nature of Quadratic Form:— Positive definite / Semi-definite / Indefinite Q.F's ?—

Let $x_1, x_2, \ldots, x_n$ be a set of complex variables & the matrix $A = [a_{ij}]$ be a Hermitian matrix of order n (i.e. $(\bar{A})^T = A$)
Then the Q.F. defined by

$$Q = X^{(*)} A X = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}\, \bar{x}_i\, x_j$$

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where the coeff^{n}: a_{ij} are complex no's with $a_{ij} = \overline{a_{ji}}$ is called a Hermitian Quadratic form.

A real or Hermitian Q.F. is said to be

i) Positive definite if $Q > 0$, $X \neq 0$ (null matrix)
$Q = 0$, $X = 0$ (null matrix)

ii) Negative definite if $Q < 0$, $X \neq 0$ (null)
$Q = 0$, $X = 0$ (null)

iii) Positive semi definite if
$Q \geq 0$, $X \neq 0$ (null)

iv) Negative semi definite if
$Q \leq 0$, $X \neq 0$ (null)

v) Indefinite if

$$\left.\begin{array}{l} Q > 0 \\ Q = 0 \\ Q < 0 \end{array}\right\} \text{ when } X \neq 0 \text{ (null)}$$

Example :— Consider
$Q = 2x_1^2 + 3x_2^2$ that is +ive definite. The Q.F. given by
$Q = -ax_1^2 - bx_2^2$ $(a > 0, b > 0)$
is −ive definite Q.F.

On the other hand
The Q.F. is given by
$Q = (x_1 - 2x_2)^2 + 3x_3^2$ is +ive

Semi definite form.

$\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$ is not null matrix.

Similarly the Q.F. is given by

$Q = -(x_1 - 2x_2)^2 - 3x_3^2$ is a negative semi definite.

$Q = (x_1 - 2x_2)^2 - 3x_3^2$ is a indefinite Q.F.

Thm :- If A is any real symmetric matrix then $\exists$ an orthogonal matrix P such that P^TAP is a diagonal matrix with real elements.

Proof: Let $A = [a_{ij}]$ be a real square symmetric matrix of $X_1, X_2, \ldots, X_n$ are the eigen vectors corresponds to the eigen values $\lambda_1, \lambda_2, \ldots, \lambda_n$ respectively then we can write

$AX_i = \lambda_i X_i$ --- ① where $i = 1, 2, \ldots, n$

suppose B denotes the square matrix whose columns are the vectors $X_1, X_2, \ldots, X_n$ so that $B = [X_1, X_2, \ldots, X_n]$ --- ②

then

$$AB = A[X_1, X_2, \ldots, X_n] = [AX_1, AX_2, \ldots, AX_n]$$
$$= [\lambda_1 X_1, \lambda_2 X_2, \ldots, \lambda_n X_n]$$
$$= BD \text{ --- ③}$$

where $D =$ diagonal matrix $[\lambda_1, \lambda_2, \ldots, \lambda_n]$

Let $Z_1, Z_2, \ldots, Z_n$ be the vectors which are obtain a group from the vectors $X_1, X_2, \ldots, X_n$ by Gram Schmidt or the generalisation process & let P denote the matrix.

$$P = [Z_1, Z_2, \ldots, Z_n]$$

it follows then that P is orthogonal satisfying the condition $P^TP = I$ from eq^n (3), on the pre multiplying by B^{-1} on both side we have

$$B^{-1}AB = D \quad \text{---} (4)$$

But the matrix B is converted to the orthogonal matrix P by orthogonalization process. so (4) becomes

$$P^TAP = D \quad \text{because } P^TP = I$$

Orthogonal Reduction of Quadratic Forms:

Let X^TAX be a real quadratic form & suppose $\lambda_1, \lambda_2, \ldots \lambda_n$ are the eigen values of the charateristic eq^n.

$|A - \lambda I| = 0$, then their exist a real orthogonal transformation $X = PY$ which transforms the X' variably into Y variables such that

$X^T A X = (PY)^T A (PY) = Y^T (P^T A P) Y$
$= Y^T \cdot \text{diag.} [\lambda_1, \lambda_2, \ldots, \lambda_n] Y$
$= \lambda_1 y_1^2 + \lambda_2 y_2^2 + \cdots + \lambda_n y_n^2$

Q. Find the orthogonal transformation which will transform the quadratic form
$x^2 + 5y^2 + 3z^2 + 8yz - 8zx$, into sum of squares. find the reduced form.

Sol. The matrix of the Q.F. is given by

$$A = \begin{bmatrix} 1 & 0 & -4 \\ 0 & 5 & 4 \\ -4 & 4 & 3 \end{bmatrix}$$

The chr eqn is given by

$$|A - \lambda I| = \begin{bmatrix} 1-\lambda & 0 & -4 \\ 0 & 5-\lambda & 4 \\ -4 & 4 & 3-\lambda \end{bmatrix} = 0$$

or $(1-\lambda)[(5-\lambda)(3-\lambda) - 16] - 4[0 - (5-\lambda)(-4)] = 0$

or $(1-\lambda)[15 - 8\lambda + \lambda^2 - 16] - 4(20 - 4\lambda) = 0$

$\Rightarrow -1 - 8\lambda + \lambda^2 + \lambda + 8\lambda^2 - \lambda^3 - 80 + 16\lambda = 0$

$\Rightarrow -\lambda^3 + 9\lambda^2 + 9\lambda - 81 = 0$

$\Rightarrow \lambda^3 - 9\lambda^2 - 9\lambda + 81 = 0$

at $\lambda = 3$

$\because\ 3 \,|\ 1 \quad -9 \quad -9$
$\quad\quad 27 \quad -81 \quad -27$

$$(A + 3I)X = 0$$ $A - \lambda I$

$$\left| \begin{matrix} 1 & 0 & -4 \\ 0 & 5 & 4 \\ -4 & 4 & 3 \end{matrix} \right| + 3 \left| \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix} \right|$$

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The eigen vector corresponding to $\lambda = -3$ is given by

$-2x_1 - 4x_3 = 0$
$2x_2 + 4x_3 = 0$
$-4x_1 + 4x_2 = 0$

$$\begin{cases} 4x_1 + 0\cdot x_2 - 4x_3 = 0 \\ 0\cdot x_1 + 8x_2 + 4x_3 = 0 \\ -4x_1 + 4x_2 + 6x_3 = 0 \end{cases}$$

$$\frac{x_1}{48 - 16} = \frac{x_2}{-16 - 0} = \frac{x_3}{0 + 32}$$

$$\frac{x_1}{32} = \frac{x_2}{-16} = \frac{x_3}{32}$$

$$\frac{x_1}{2} = \frac{x_2}{-1} = \frac{x_3}{2}$$

Thus corresponding to $\lambda = -3$ the eigen vector is

$$X_2 = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$$

Finally the eigen vector corresponding to $\lambda = 9$

$$\begin{cases} -8x_1 + 0\cdot x_2 - 4x_3 = 0 \\ 0\cdot x_1 - 4x_2 - 4x_3 = 0 \\ -4x_1 + 4x_2 - 6x_3 = 0 \end{cases}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \\ \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \\ -\frac{1}{3} & \frac{2}{3} & \frac{2}{3} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

i.e. the orthogonal transformation is

$$x_1 = \frac{1}{3}(2y_1 + 2y_2 - y_3)$$

$$x_2 = \frac{1}{3}(2y_1 - y_2 + 2y_3)$$

$$x_3 = \frac{1}{3}(-y_1 + 2y_2 + 2y_3)$$

this will reduce the given quadratic form

$X^T A X$ to $\underline{Y^T \, dig(+3, -3, 9) \, Y}$

which is equal to

$$+3y_1^2 - 3y_2^2 + 9y_3^2$$