Multivariate Normal Distribution :-

The density function of a multivariate normal distn. has an analogous form to that of a univariate normal distn.

Now, let

$$\underline{X}_{p\times 1} = \begin{bmatrix} x_1 \\ \vdots \\ x_p \end{bmatrix}_{p\times 1}$$

and $x_1, x_2, \ldots, x_p$ are p-indept. normal variate then the joint p.d.f. of $x_1, x_2, \ldots, x_p$ is given by

$$f(x_1, x_2, \ldots, x_p) = \prod_{i=1}^{p} \frac{1}{\sqrt{2\pi}\,\sigma_i} e^{-\frac{1}{2}\left(\frac{x_i-\mu_i}{\sigma_i}\right)^2}$$

$$= \frac{1}{(2\pi)^{p/2}\,\sigma_1\sigma_2\cdots\sigma_p} e^{-\frac{1}{2}\sum_{i=1}^{p}\left(\frac{x_i-\mu_i}{\sigma_i}\right)^2} \quad \text{—— (1)}$$

$$E(\underline{X}) = \underline{\mu}$$

$$\Sigma = \begin{bmatrix} \sigma_1^2 & 0 & \cdots & 0 \\ 0 & \sigma_2^2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \sigma_p^2 \end{bmatrix}$$

$$\Rightarrow \Sigma^{-1} = \text{diag.}\left[\frac{1}{\sigma_1^2} \cdots \frac{1}{\sigma_p^2}\right]$$

Consider $(\underline{X}-\underline{\mu})'\,\Sigma^{-1}\,(\underline{X}-\underline{\mu})$

$$= [(x_1-\mu_1) \cdots (x_p-\mu_p)] \begin{bmatrix} \frac{1}{\sigma_1^2} & 0 & \cdots & 0 \\ 0 & \frac{1}{\sigma_2^2} & \cdots & 0 \\ \vdots & \vdots & & \\ 0 & 0 & \cdots & \frac{1}{\sigma_p^2} \end{bmatrix} \begin{bmatrix} x_1-\mu_1 \\ x_2-\mu_2 \\ \vdots \\ x_p-\mu_p \end{bmatrix}$$

① $\therefore (\underline{X}-\underline{\mu})' \Sigma^{-1} (\underline{X}-\underline{\mu}) = \sum_{i=1}^{p} \left(\frac{x_i - \mu_i}{\sigma_i} \right)^2$ ②

also $|\Sigma|^{1/2} = \sigma_1 \sigma_2 \cdots \sigma_p$

substituting these value in ①, we have

$$f(\underline{X}) = \frac{1}{(2\pi)^{p/2}} \frac{e^{-\frac{1}{2}(\underline{X}-\underline{\mu})'\Sigma^{-1}(\underline{X}-\underline{\mu})}}{|\Sigma|^{1/2}}$$

This is a p.d.f. of multivariate normal distribution (Indept. case) follows $N_p(\underline{\mu}, \Sigma)$

Taking $p = 1 \Rightarrow N_1(\mu_i, \sigma_i)$

Derivation of Multivariate Normal Distribution :-

Case :- When the components of a random vector are not indept. i.e. a non i.i.d. case.

In general analogous to eqn no. ① we may write

$$f(\underline{X}) = k e^{-\frac{1}{2}(\underline{X}-\underline{B})' A (\underline{X}-\underline{B})}$$

where k : is a constant (Normalising)

$\underline{B}$: is a vector (of order $p \times 1$)

& A : is a p.s.d. (positive semi definite) symmetric matrix (of order $p \times p$)

Determination of k :-

Let $(\underline{X}-\underline{B}) = C\underline{Y}$

where C is any NSM (Non-singular matrix) and it is determined by using the result that if A is a p.s.d. matrix then there exists a NSM C s.t.

$$C'AC = I \Rightarrow |C'||A||C| = |I| = 1$$

since $(\underline{X} - \underline{B}) = C\underline{Y}$ $\qquad |J| = |C|$

as $f(\underline{X}) = k\, e^{-\frac{1}{2}(\underline{X}-\underline{B})' A (\underline{X}-\underline{B})}$

we have

$$f(\underline{Y}) = k\, e^{-\frac{1}{2}\underline{Y}'C'AC\underline{Y}} \cdot |J|$$

$$= k\, e^{-\frac{1}{2}\underline{Y}'\underline{Y}} \cdot |J|$$

$$= k\, e^{-\frac{1}{2}\sum_{i=1}^{p} y_i^2} \cdot |C|$$

i.e. $\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} k\, e^{-\frac{1}{2}\sum_{i=1}^{p} y_i^2}\, dy_1\, dy_2 \cdots dy_p \cdot |C| = 1$

or $k\,|C| \prod_{i=1}^{p} \left\{ \int_{-\infty}^{\infty} e^{-\frac{1}{2}y_i^2}\, dy_i \right\} = 1$

[as we know that $N(0,1) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} = 1$

$\Rightarrow \int_{-\infty}^{\infty} e^{-y^2/2} dy = \sqrt{2\pi}$]

$\therefore$ $k\,|C| \prod_{i=1}^{p} (\sqrt{2\pi}) = 1$

or $k\,|C| (\sqrt{2\pi})^p = 1$

$$k = \frac{1}{(2\pi)^{p/2}} \times \frac{1}{|C|}$$

or $k = \frac{1}{(2\pi)^{p/2}} \cdot \frac{1}{1/|A|^{1/2}}$ $\qquad \because C'AC = I$

$|C'||A||C| = I$

$|C|^2 |A| = I$

$\therefore k = \frac{|A|^{1/2}}{(2\pi)^{p/2}}$ $\qquad |C| = \frac{1}{|A|^{1/2}}$

as $|C'| = |C|$

Determination of B :-

Since $(\underline{X} - \underline{B}) = C\underline{Y}$

$\Rightarrow \quad \underline{X} = \underline{B} + C\underline{Y}$

or $\quad E(\underline{X}) = \underline{B} + C\,E(\underline{Y})$

Now $f(\underline{Y}) = \dfrac{1}{(2\pi)^{p/2}} e^{-\frac{1}{2}\underline{Y}'\underline{Y}} = \dfrac{1}{(2\pi)^{p/2}} e^{-\frac{1}{2}\sum_{i=1}^{p} y_i^2}$

$\therefore E(y_i) = \int_{-\infty}^{\infty} y_i f(y_i)\, dy_i$

$= \int_{-\infty}^{\infty} y_i \dfrac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y_i^2} dy_i \prod_{\substack{j=1 \\ (i \neq j)}}^{p} \left\{ \int_{-\infty}^{\infty} \dfrac{1}{(2\pi)^{\frac{p-1}{2}}} e^{-\frac{1}{2}\sum_{j=1}^{p} y_j^2} dy_j \right\}$

$= 0 \times 1 = 0 \qquad \left[\because E(y_i) = 0 \text{ where } y_i \sim N(0,1)\right]$

i.e. $E(y_i) = 0 \qquad$ [as $y_i e^{-\frac{1}{2}y_i^2}$ is an odd funᶜᵗ of y_i. Thus integral value is zero

$\Rightarrow \quad E(\underline{Y}) = 0$

$\therefore E(\underline{X}) = \underline{B} + C\,E(\underline{Y})$

$\underline{\mu} = \underline{B} \qquad$ as $E(\underline{X}) = \underline{\mu}$

$\therefore \Rightarrow \underline{B} = \underline{\mu}$

Determination of A :-

We know that

$\Sigma = E(\underline{X} - \underline{\mu})(\underline{X} - \underline{\mu})'$

$= E(C\underline{Y}\,\underline{Y}'C')$

or $\quad \Sigma = C\,E(\underline{Y}\,\underline{Y}')\,C'$

Let us consider the joint moment

$$E(y_j y_k) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} y_j y_k \frac{1}{(2\pi)^{p/2}} e^{-\frac{1}{2}\underline{y}'\underline{y}} \, dy_1 \cdots dy_p$$

(by def^n)

$$= \left\{ \int_{-\infty}^{\infty} y_j \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y_j^2} dy_j \right\} \left\{ \int_{-\infty}^{\infty} y_k \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y_k^2} \right\}$$

$$\times \prod_{\substack{l=1 \\ l \neq j \neq k}}^{p} \left\{ \int_{-\infty}^{\infty} \frac{1}{(2\pi)^{\frac{p-2}{2}}} e^{-\frac{1}{2}y_l^2} dy_l \right\}$$

$= 0$ as they are odd fun^n.

$\therefore E(y_j y_k) = 0$ for $j \neq k$

Now let $j = k$, then

$$E(y_j^2) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} y_j^2 \frac{1}{(2\pi)^{p/2}} e^{-\frac{1}{2}\sum_{j=1}^{p} y_j^2} dy_1 \cdots dy_p$$

$$= \int_{-\infty}^{\infty} y_j^2 \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y_j^2} dy_j \cdot \prod_{\substack{k=1 \\ k \neq j}}^{p} \left\{ \int_{-\infty}^{\infty} \frac{1}{(2\pi)^{\frac{p-1}{2}}} e^{-\frac{1}{2}y_k^2} dy_k \right\}$$

$= 1 \times 1$

$= 1$

$$\boxed{\begin{aligned} & y_j \sim N(0,1) \\ & E(y_j) = 0, \; V(y_j) = 1 \\ & V(y_j) = E(y_j^2) - [E(y_j)]^2 \\ & 1 = E(y_j^2) \end{aligned}}$$

$$\therefore E(y_j y_k) = \begin{cases} 0 & \text{for } j \neq k \\ 1 & \text{for } j = k \end{cases}$$

$\therefore$ Var. Cov. Matrix of $\underline{Y} = E[\underline{Y} - E(\underline{Y})][\underline{Y} - E(\underline{Y})]'$

$= E(\underline{Y}\underline{Y}') = E(\underline{Y}'\underline{Y}) = I_p$

$$c^{-1}(\underline{X}-\underline{\mu}) = \underline{Y}$$

Again $(\underline{X}-\underline{B}) = c\underline{Y} \Rightarrow$

$\therefore E[c^{-1}(\underline{X}-\underline{\mu})(\underline{X}-\underline{\mu})'(c^{-1})'] = I$

or $c^{-1}E(\underline{X}-\underline{\mu})(\underline{X}-\underline{\mu})'(c')^{-1} = I$

or $c^{-1}\Sigma(c')^{-1} = I$

or $\Sigma = cc'$ —— (i)

also $c'Ac = I$

or $A = (c')^{-1} I c^{-1}$

or $A = (cc')^{-1}$ (ii)

(i) & (ii) $\Rightarrow A = \Sigma^{-1}$

now putting the value of A in k

$$k = \frac{|A|^{1/2}}{(2\pi)^{p/2}} = \frac{(\Sigma^{-1})^{1/2}}{(2\pi)^{p/2}} = \frac{1}{(2\pi)^{p/2}|\Sigma|^{1/2}}$$

Now $f(\underline{X}) = k\, e^{-\frac{1}{2}(\underline{X}-\underline{\mu})'A(\underline{X}-\underline{\mu})}$

$$f(\underline{X}) = \frac{1}{(2\pi)^{p/2}|\Sigma|^{1/2}} \times e^{-\frac{1}{2}(\underline{X}-\underline{\mu})'\Sigma^{-1}(\underline{X}-\underline{\mu})}$$

$\Rightarrow \underline{X} \sim N_p(\mu, \Sigma)$

where $\underline{X}$ is a random vector

Special case :— $p = 2$

$$\underline{X} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}_{2\times1}, \quad \underline{\mu} = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}_{2\times1}$$

$$\Sigma = \begin{bmatrix} Var(X_1) & cov(X_1, X_2) \\ Cov(X_2, X_1) & Var(X_2) \end{bmatrix}_{2\times 2}$$

$$= \begin{bmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{21} & \sigma_2^2 \end{bmatrix}_{2\times 2}$$

or $\Sigma = \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix}$ as $cov(X_1, X_2) = cov(X_2, X_1)$

$\Rightarrow cov(X_1, X_2) = \rho\sigma_1\sigma_2$

$$\Sigma^{-1} = \begin{bmatrix} \frac{1}{\sigma_1^2} & \frac{1}{\rho\sigma_1\sigma_2} \\ \frac{1}{\rho\sigma_1\sigma_2} & \frac{1}{\sigma_2^2} \end{bmatrix}$$

$$\Sigma^{-1} = \frac{1}{|\Sigma|} \cdot I$$

$$\Sigma^{-1} = \frac{1}{\sigma_1^2\sigma_2^2(1-\rho^2)} \begin{bmatrix} \sigma_2^2 & -\rho\sigma_1\sigma_2 \\ -\rho\sigma_1\sigma_2 & \sigma_1^2 \end{bmatrix}$$

and

$$|\Sigma| = \sigma_1^2\sigma_2^2 - \rho^2\sigma_1^2\sigma_2^2 = \sigma_1^2\sigma_2^2(1-\rho^2)$$

$$|\Sigma|^{1/2} = \sigma_1\sigma_2\sqrt{(1-\rho^2)}$$

$$(\underline{x}-\underline{\mu})'\Sigma^{-1}(\underline{x}-\underline{\mu}) = [(x_1-\mu_1) \quad (x_2-\mu_2)] \frac{1}{\sigma_1^2\sigma_2^2(1-\rho^2)} \begin{bmatrix} \sigma_2^2 & -\rho\sigma_1\sigma_2 \\ -\rho\sigma_1\sigma_2 & \sigma_1^2 \end{bmatrix} \times \begin{bmatrix} (x_1-\mu_1) \\ (x_2-\mu_2) \end{bmatrix}$$

$$= \frac{1}{(1-\rho^2)}\left[\frac{(x_1-\mu_1)^2}{\sigma_1^2} - \frac{2\rho(x_1-\mu_1)(x_2-\mu_2)}{\sigma_1\sigma_2} + \frac{(x_2-\mu_2)^2}{\sigma_2^2}\right]$$

$$\therefore f(\underline{x}) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{(1-\rho^2)}} e^{-\frac{1}{2(1-\rho^2)}\left[\frac{(x_1-\mu_1)^2}{\sigma_1^2} - \frac{2\rho(x_1-\mu_1)(x_2-\mu_2)}{\sigma_1\sigma_2} + \frac{(x_2-\mu_2)^2}{\sigma_2^2}\right]}$$

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which is a Bivariate Normal distribution (BVND) with parameters $(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \rho)$

Note :- Number of parameter in a multivariate normal distribution

$$p + \frac{p(p+1)}{2} \qquad \left[\text{eg. } p = 2 \quad 2 + \frac{2(2+1)}{2} = 5 \right]$$

Ex: If $\underline{X} = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}_{3\times 1}$ & $\underline{\mu} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}_{3\times 1}$

& $\Sigma = \begin{bmatrix} 4 & 1 & 2 \\ 1 & 9 & 3 \\ 2 & 3 & 16 \end{bmatrix}$

then find out $f(\underline{X}) = ?$

Properties of Multivariate Normal Distribution :-

1) Invariance Property of MVND :-

If $\underline{X} \sim N_p(\mu, \Sigma)$ then

$\underline{Y} = C\underline{X}$ where C is a NSM, is distributed as $\underline{Y} \sim N_p(C\underline{\mu}, C\Sigma C')$

proof:-

As $\underline{Y} = C\underline{X}$ is a ~~NSM~~ non-singular transformation

$\underline{X} = C^{-1}\underline{Y}$ and for this type of transformation

$|J| = |C^{-1}|$

As

$$f(\underline{X}) = \frac{1}{(2\pi)^{p/2}} \frac{1}{|\Sigma|^{1/2}} \times e^{-\frac{1}{2}(\underline{X}-\underline{\mu})'\Sigma^{-1}(\underline{X}-\underline{\mu})}$$

we have

$$f(\underline{Y}) = \frac{1}{(2\pi)^{p/2}} \frac{1}{|\Sigma|^{1/2}} e^{-\frac{1}{2}(C^{-1}\underline{Y}-\underline{\mu})'\Sigma^{-1}(C^{-1}\underline{Y}-\underline{\mu})}$$

$$\times \; |C^{-1}|$$

$$|C^{-1}| = |C|^{-1/2} \; |C'|^{-\frac{1}{2}}$$

$$f(\underline{Y}) = \frac{1}{(2\pi)^{p/2}} \frac{1}{|C|^{1/2} \; |\Sigma|^{1/2} \; |C'|^{1/2}} \times e^{-\frac{1}{2}\{C^{-1}(\underline{Y}-C\underline{\mu})\}'\Sigma^{-1}\{C^{-1}(\underline{Y}-C\underline{\mu})\}}$$

or $$f(\underline{Y}) = \frac{1}{(2\pi)^{p/2} |C\Sigma C'|^{1/2}} e^{-\frac{1}{2}(\underline{Y}-C\underline{\mu})'(C^{-1})'\Sigma^{-1}C^{-1}(\underline{Y}-C\underline{\mu})}$$

$$= \frac{1}{(2\pi)^{p/2} |C\Sigma C'|^{1/2}} e^{-\frac{1}{2}(\underline{Y}-C\underline{\mu})'(C\Sigma C')^{-1}(\underline{Y}-C\underline{\mu})}$$

$$\therefore \; \underline{Y} \sim N_p(C\underline{\mu}, \; C\Sigma C')$$

Examples :-

Ex1 $\Rightarrow$ find out mean, variance and correlation if $f(\underline{X}) = \frac{1}{(2\pi)} \exp\left[-\frac{1}{2}\{(x-1)^2 + (y-2)^2\}\right]$ —— (*)

Solution :- The standard form of MVND is

$$f(\underline{X}) = \frac{1}{(2\pi)^{p/2} |\Sigma|^{1/2}} e^{-\frac{1}{2}(\underline{X}-\underline{\mu})'\Sigma^{-1}(\underline{X}-\underline{\mu})} \quad \text{—— (1)}$$

comparing (*) with (1) we observe

$$|\Sigma|^{1/2} = 1 \Rightarrow |\Sigma| = 1 \Rightarrow \Sigma = I$$

$$\Sigma = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow \begin{array}{l} var(X) = 1 \\ var(Y) = 1 \end{array}$$

$$(\underline{X} - \underline{\mu})' \Sigma^{-1} (\underline{X} - \underline{\mu}) = \begin{bmatrix} X - \mu_x & Y - \mu_y \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} X - \mu_x \\ Y - \mu_y \end{bmatrix}$$

$$= (X - \mu_x)^2 + (Y - \mu_y)^2$$ (comparing with *)

$$\Rightarrow \mu_x = 1, \quad \mu_y = 2$$

$$\Rightarrow \underline{\mu} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \text{ and } \sigma_x^2 = \sigma_y^2 = 1$$

Now for bivariate dist^n.

$$|\Sigma|^{1/2} = \sigma_1 \sigma_2 \sqrt{(1 - \rho^2)}$$

$$\sigma_1 \sigma_2 \sqrt{(1 - \rho^2)} = 1$$

$$\Rightarrow (1 - \rho^2) = 1$$

$$\Rightarrow \rho^2 = 0$$

$$\Rightarrow \rho = 0$$

$$\therefore \Sigma = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} ; \quad \mu = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\rho = 0$$

Ex 2:- $f(X, Y) = \frac{1}{2 \cdot 4 \cdot \pi} \exp\left\{ - \frac{\frac{x^2}{4} - 1.6\, xy + y^2}{(0.72)} \right\}$

obtain μ, Σ, ρ

comparing (*) with (1) we observe

$|\Sigma|^{1/2} = 1 \Rightarrow |\Sigma| = 1 \Rightarrow \Sigma = I$

$\Sigma = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow \begin{array}{l} var(X) = 1 \\ var(Y) = 1 \end{array}$

$(\underline{X} - \underline{\mu})' \Sigma^{-1} (\underline{X} - \underline{\mu}) = [X - \mu_x \quad Y - \mu_y] \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} X - \mu_x \\ Y - \mu_y \end{bmatrix}$

$= (X - \mu_x)^2 + (Y - \mu_y)^2$ (comparing with *)

$\Rightarrow \mu_x = 1, \quad \mu_y = 2$

$\Rightarrow \underline{\mu} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\sigma_x^2 = \sigma_y^2 = 1$

Now for bivariate distn.

$|\Sigma|^{1/2} = \sigma_1 \sigma_2 \sqrt{(1-\rho^2)}$

$\sigma_1 \sigma_2 \sqrt{(1-\rho^2)} = 1$

$\Rightarrow (1-\rho^2) = 1$

$\Rightarrow \rho^2 = 0$

$\Rightarrow \rho = 0$

$\therefore \Sigma = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} ; \mu = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$\rho = 0$

Ex. 2:- $f(x, y) = \frac{1}{2.4\pi} \exp\left\{ -\frac{\frac{x^2}{4} - 1.6xy + y^2}{(0.72)} \right\}$

obtain μ, Σ, ρ

Ex 3 — $f(\underline{x}) = \frac{1}{2\pi} \exp\left\{-\frac{1}{2}\left(x^2 + y^2 - [illegible]x - 6y + 3\right)\right\}$

obtain μ, Σ & ρ

Properties (continue)

2) Characteristic Function :—

If $\underline{x} \sim N_p(\mu, \Sigma)$, $\phi_{\underline{x}}(\underline{t}) = ?$

by def^{n}.

$$\phi_{\underline{x}}(t) = E(e^{i\underline{t}'\underline{x}}) \quad \text{where } \underline{t} \text{ is any real vector}$$

$[\ \phi_x(t) = E(e^{itx})\ ; \quad x \sim N(\mu, \sigma^2)\ ]$

We assume that Σ is a NSM and hence there exists another NSM C, s.t

$$C'\Sigma^{-1}C = I_p \Rightarrow \Sigma^{-1} = (C')^{-1}C^{-1} = (CC')^{-1}$$

$$\Rightarrow \Sigma = CC'$$

consider the transformation $(\underline{X} - \underline{\mu}) = C\underline{Y}$

$$\Rightarrow \underline{X} = \underline{\mu} + C\underline{Y}$$

We know that $E(\underline{X} - \underline{\mu}) = E(\underline{X}) - \underline{\mu} = 0$

$$\Rightarrow E(\underline{Y}) = 0$$

again the VCM of $\underline{Y}$ is

$$E(\underline{Y}\underline{Y}') = I_p$$

$$\Rightarrow \underline{Y} \sim N_p(0, I_p)$$

$$\Rightarrow y_j \sim N(0,1) \quad (\text{any } j^{th} \text{ component of } \underline{Y}) \sim N(0,1)$$

since $\phi_{\underline{X}}(t) = E(e^{i\underline{t}'\underline{X}})$

$$= E[e^{i\underline{t}'(\underline{\mu} + C\underline{Y})}]$$

$$= E[e^{i\underline{t}'\underline{\mu}}\, e^{i\underline{t}'C\underline{Y}}]$$

$$\Rightarrow \quad \phi_{\underline{X}}(t) = e^{i\underline{t}'\underline{\mu}} \; E\left(e^{i\underline{t}'C\underline{Y}}\right)$$

let $\underline{t}'C = U'$ (vector)

$$\phi_{\underline{X}}(t) = e^{i\underline{t}'\underline{\mu}} \; E\left(e^{iU'\underline{Y}}\right)$$

$$= e^{i\underline{t}'\underline{\mu}} \; E\left(e^{i\sum_{j=1}^{p} u_j Y_j}\right)$$

$$= e^{i\underline{t}'\underline{\mu}} \prod_{j=1}^{p} E\left(e^{iu_j Y_j}\right)$$

$$= e^{i\underline{t}'\underline{\mu}} \; e^{-\frac{1}{2}U'U}$$

$$= e^{i\underline{t}'\underline{\mu}} \; e^{-\frac{1}{2}\underline{t}'CC'\underline{t}} \quad \text{as } CC' = \Sigma$$

$$\boxed{\phi_{\underline{X}}(t) = e^{i\underline{t}'\underline{\mu}} \; e^{-\frac{1}{2}\underline{t}'\Sigma\underline{t}}}$$

$\underline{t}'_{1\times p} \, C_{p\times p} = (t'C)_{1\times p}$

Now if

$U_{p\times 1} \Rightarrow \underline{U}'_{1\times p}$

$\therefore (\underline{t}'C)_{1\times p} = \underline{U}'_{1\times p}$

$E(e^{iUY}) = \phi_Y(U)$

For $N(\mu, \sigma^2)$

$\phi_X(t) = e^{\mu t - \frac{1}{2}\sigma^2 t^2}$

for $N(0,1)$

$\phi_Z(t) = e^{-t^2/2}$

$E[e^{iUY}] = e^{-\frac{1}{2}t^2}$

Here

$$(i\underline{t}'\underline{\mu})_{1\times 1} = i\,t'_{1\times p}\,\mu_{p\times 1}$$

$$= \text{a scalar}$$

$$\underline{t}'\Sigma\underline{t} = \underline{t}'_{1\times p}\,\Sigma_{p\times p}\,\underline{t}_{p\times 1} = \text{a scalar}$$

both quantity $i\underline{t}'\underline{\mu}$ and $\underline{t}'\Sigma\underline{t}$ are scalar having same order so subtraction is possible

Hence $\phi_{\underline{X}}(t) = e^{i\underline{t}'\underline{\mu} - \frac{1}{2}\underline{t}'\Sigma\underline{t}}$

is characteristic function of MVND.

3) If $X_1, X_2, \ldots X_p$ have a joint MND then a necessary and sufficient condition (NASC) that one subset $X^{(1)}$ is independent of the other subset $X^{(2)}$, consisting the remaining variables, is that each covariance of a variable from $X^{(1)}$ and a variate from $X^{(2)}$ be zero.

Or in other words let $X^{(1)}$ and $X^{(2)}$ be two subsets of X having MND, the mean vector and the VCM are accordingly partitioned as

$$\underline{\mu} = \begin{bmatrix} \underline{\mu}^{(1)} \\ \underline{\mu}^{(2)} \end{bmatrix};$$

$$\Sigma = \begin{bmatrix} \Sigma_{11\,q\times q} & \Sigma_{12\,q\times(p-q)} \\ \Sigma_{21\,(p-q)\times q} & \Sigma_{22\,(p-q)\times(p-q)} \end{bmatrix}$$

then $X^{(1)}$ and $X^{(2)}$ are independent iff $\Sigma_{12} = \underline{0}$

where

$$\Sigma_{11} = E[X^{(1)} - \mu^{(1)}][X^{(1)} - \mu^{(1)}]'$$

$$\Sigma_{12} = E[X^{(1)} - \mu^{(1)}][X^{(2)} - \mu^{(2)}]'$$

$$\Sigma_{11} = E[X^{(1)}_{q\times1} - \mu^{(1)}_{q\times1}][X^{(1)}_{q\times1} - \mu^{(1)}_{q\times1}]'$$

$$= (q\times1)(q\times1)' = (q\times1)(1\times q)$$

$$= q\times q$$

Proof :-

Necessary Part :- Let $X^{(1)}$ and $X^{(2)}$ be any two subsets if X having MND, the mean vector and the VCM be accordingly partitioned as

$$\underline{\mu} = \begin{bmatrix} \underline{\mu}^{(1)} \\ \underline{\mu}^{(2)} \end{bmatrix}, \quad \Sigma = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix}$$

Let $X^{(1)}$ and $X^{(2)}$ be independent and let X_i be a variate from $\underline{X}^{(1)}$ and X_j be any other variate from $\underline{X}^{(2)}$. Then we have to show that $\text{cov}(X_i, X_j) = 0$

Now

$$\text{cov}(X_i X_j) = E[(X_i - E(X_i))(X_j - E(X_j))]$$

$$= E[(X_i - \mu_i)(X_j - \mu_j)]$$

$$\text{cov}(X_i X_j) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} (X_i - \mu_i)(X_j - \mu_j) f(\underline{X})\, dx_1 \cdots dx_p$$

Since $\underline{X}^{(1)}$ and $\underline{X}^{(2)}$ are independent, then

$$f(\underline{X}) = f(\underline{X}^{(1)})\, f(\underline{X}^{(2)})$$

$$\text{cov}(X_i X_j) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} (X_i - \mu_i)(X_j - \mu_j) f(\underline{X}^{(1)}) f(\underline{X}^{(2)})\, d\underline{x}^{(1)} d\underline{x}^{(2)}$$

$$= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} (X_i - \mu_i) f(\underline{X}^{(1)})\, d\underline{x}^{(1)} \cdot \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} (X_j - \mu_j) f(\underline{X}^{(2)})\, d\underline{x}^{(2)}$$

$$= E(X_i - \mu_i) \cdot E(X_j - \mu_j)$$

$$\text{cov}(X_i X_j) = 0 \qquad \boxed{\because E(X) = \mu}$$

for $i = 1, 2, \ldots, q$ and $j = q+1, \ldots, p$

$$\Rightarrow \Sigma_{12} = \underline{0}$$

Since if

$\Sigma_{12} = 0 \quad \therefore \Sigma_{21} = \underline{0}$ since the matrix is symmetric

Sufficiency Part :-

We know that $\Sigma = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix}$

if $\Sigma_{12} = 0$

$$\Sigma = \begin{bmatrix} \Sigma_{11} & 0 \\ 0 & \Sigma_{22} \end{bmatrix} =$$

$$|\Sigma| = |\Sigma_{11} \Sigma_{22}| = |\Sigma_{11}| \cdot |\Sigma_{22}|$$

$$\Sigma^{-1} = \begin{vmatrix} \Sigma_{11}^{-1} & 0 \\ 0 & \Sigma_{22}^{-1} \end{vmatrix}$$

$$f(\underline{X}) = \frac{1}{(2\pi)^{p/2} |\Sigma|^{1/2}} \exp\left[-\frac{1}{2}(\underline{X}-\underline{\mu})' \Sigma^{-1} (\underline{X}-\underline{\mu})\right]$$

consider the exponent part

$$(\underline{X}-\underline{\mu})' \Sigma^{-1} (\underline{X}-\underline{\mu}) = \left[(\underline{X}^{(1)} - \underline{\mu}^{(1)}) \quad (X_2^{(2)} - \mu^{(2)})\right] \begin{bmatrix} \Sigma_{11}^{-1} & 0 \\ 0 & \Sigma_{22}^{-1} \end{bmatrix} \begin{bmatrix} X^{(1)} - \mu^{(1)} \\ X^{(2)} - \mu^{(2)} \end{bmatrix}$$

$$= (\underline{X}^{(1)} - \underline{\mu}^{(1)}) \Sigma_{11}^{-1} (\underline{X}^{(2)} - \underline{\mu}^{(2)}) + (\underline{X}^{(2)} - \underline{\mu}^{(2)})' \Sigma_{22}^{-1} (\underline{X}^{(2)} - \underline{\mu}^{(2)})$$

$$f(\underline{X}) = \frac{1}{(2\pi)^{q/2} |\Sigma_{11}|^{1/2}} e^{-\frac{1}{2}(\underline{X}^{(1)} - \underline{\mu}^{(1)})' \Sigma_{11}^{-1} (\underline{X}^{(1)} - \mu^{(1)})} \times \frac{1}{(2\pi)^{\frac{p-q}{2}} |\Sigma_{22}|^{1/2}} e^{-\frac{1}{2}(\underline{X}^{(2)} - \mu^{(2)})' \Sigma_{22}^{-1} (\underline{X}^{(2)} - \mu^{(2)})}$$

$$f(\underline{x}) = f(\underline{x}^{(1)}) \cdot f(\underline{x}^{(2)})$$

$\Rightarrow$ $\underline{X}^{(1)}$ and $\underline{X}^{(2)}$ are independent

(i.e. p.d.f. is the product of the marginal densities).